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BARRY SPAIN

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ANALYTICAL QUADRICS by Dr. Spain, head of the Mathematics Department of Sir John Cass College, is intended to serve as a concise introduction to the analytical geometry of three dimensions.

The book begins by developing *ab initio* the theory of the plane, straight line, sphere, cone and cylinder, central quadrics and paraboloids in their standard forms. Next, the concept of the plane at infinity is introduced with the aid of homogeneous Cartesian co-ordinates and applied to the nature of the intersection of three planes and to the circular sections of quadrics. Subsequently, the properties of the quadric given by the general equation are studied. After this, the invariants of a quadratic form under translations and rotations are treated and applied to the classification and reduction of the quadric. This is followed by a discussion of foci, confocals and linear systems of quadrics. Finally, an introductory treatment is presented of plane coordinates leading to the general treatment of confocal systems.

The concept of the rank of a matrix has been widely used. The definition of a matrix and all the results required are summarized in the first appendix.

In order to benefit fully from the text, the reader is strongly advised by Dr. Spain to solve the examples for himself. However, a second appendix containing worked-out solutions or hints is provided.

This authoritative volume will undoubtedly be given the same enthusiastic reception as Dr. Spain's previous volume in the series, *Analytical Conics*.

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Head of The Mathematics Department
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PREFACE

THIS textbook is intended to serve as a *concise* introduction to the analytical geometry of three dimensions. Accordingly the same general plan of 'Analytical Conics' is adopted.

The book begins by developing *ab initio* the theory of the plane, straight line, sphere, cone and cylinder, central quadrics and paraboloids in their standard forms. Next, the concept of the plane at infinity is introduced with the aid of homogeneous Cartesian coordinates and applied to the nature of the intersection of three planes and to the circular sections of quadrics. Subsequently the properties of the quadric given by the general equation are studied. After this the invariants of a quadratic form under translations and rotations are treated and applied to the classification and reduction of the quadric. This is followed by a discussion of foci, confocals and linear systems of quadrics. Finally, an introductory treatment is presented of plane-coordinates leading to the general treatment of confocal systems.

The theory of general homogeneous coordinates has been omitted as little progress can be made in three dimensions within the compass of a short chapter.

The concept of the rank of a matrix has been widely used. The definition of a matrix and all the results required are summarised in the first Appendix.

In order to derive the full benefit of the text, the reader is strongly advised to solve the examples for himself. However, a second Appendix containing worked-out solutions or hints is provided.

My thanks are due to the Board of Trinity College, University of Dublin, for permission to include many examples selected from the University examination papers. I am indebted to Mr. P. J. Wakely for a critical reading of the manuscript and to Mr. A. C. Newell who checked the examples. For help in reading the proofs I wish to thank my wife and elder son.

Finally, I wish to express my appreciation to the staff of Messrs. Pergamon Press for their cooperation, help with the diagrams and high quality of their work.

B. S.

Sir John Cass College,
London
June, 1960

CONTENTS

I. INTRODUCTION

PAGE

1. Point coordinates 1
2. Distance between two points 2
3. Joachimsthal's section formula 3
4. Direction-cosines 4
5. Direction-ratios 5
6. Angle between two straight lines 5
7. Projections 7

II. PLANE—STRAIGHT LINE

8. Equation of a plane 9
9. Equations of a straight line 10
10. Straight line and plane 12
11. Sign of the expression $u = \lambda x + \mu y + \nu z + \rho$ 13
12. Distance from a point to a plane 14
13. Pair of planes 15
14. Bundle of planes 15
15. Shortest distance between two skew lines 16
16. Coplanar lines 17
17. Rectangular axes related to two skew lines 18

III. SPHERE

18. Equation of sphere 20
19. Conjugate points 21
20. Pole and polar 23
21. Tangent plane and normal line 23
22. Angle of intersection of two spheres 25
23. Radical plane 25
24. Coaxal spheres 26

IV. QUADRIC CONE—CYLINDER

25. Quadric cone 29
26. Cone with vertex at the origin 29
27. Circular cone 30
28. Conditions that $\psi(x,y,z)$ be a circular cone 31
29. Angle between two generators 31
30. Reciprocal cone 33
31. Rectangular and orthogonal cones 34
32. Quadric cylinder 34

V. CENTRAL QUADRIC

33. Central quadric 36
34. Ellipsoid 37
35. Hyperboloid of one sheet 38
36. Hyperboloid of two sheets 39
37. Polar properties 39
38. Generators 41

PAGE
39. Generators of hyperboloid of one sheet 42
40. Normals 43
41. Centre of a section 44
42. Axes of central plane section 45
43. Axes of non-central plane section 46

VI. PARABOLOID

44. Paraboloid 48
45. Polar properties 50
46. Generators 50
47. Generators of hyperbolic paraboloid 51
48. Normals 52
49. Centre of a section 52
50. Axes of plane section 53

VII. HOMOGENEOUS COORDINATES

51. Homogeneous coordinates 55
52. Point at infinity 55
53. Straight line 55
54. Plane 57
55. Intersection of three planes 58
56. Circle at infinity 59
57. Circular sections of central quadric 61
58. Paraboloid 62

VIII. GENERAL QUADRIC

59. Quadric 65
60. Cofactor notation 66
61. Conjugate points 66
62. Pole and polar 67
63. Tangent plane 68
64. Enveloping cone 68
65. Generators 69
66. Polar of a line 70
67. Conjugate lines 71
68. Conjugate diameters 71
69. Principal diametral planes 73
70. Discriminating cubic 74
71. Circular sections 75
72. Rank of matrix Δ 76

IX. TRANSFORMATION OF AXES—INVARIANTS

73. Translation of axes 78
74. Invariants of a quadratic form under translation 79
75. Rotation of axes 80
76. Invariants of quadratic form under rotation 82
77. General transformation of coordinates 83

X. CLASSIFICATION AND REDUCTION OF QUADRIC

78. Equation of quadric 84
79. Classification of quadrics 85
80. Central quadric and cone 87

PAGE
81. Paraboloid 88
82. Elliptic and hyperbolic cylinder 89
83. Parabolic cylinder and parallel plane-pair 90

XI. FOCI—CONFOCALS

84. Focus 93
85. Foci of central quadric 94
86. Foci of paraboloid 96
87. Confocal central quadrics 97
88. Confocal paraboloids 98

XII. LINEAR SYSTEMS OF QUADRICS

89. Pencil of quadrics 101
90. Self-polar tetrahedron of a pencil of quadrics 102
91. Contact of quadrics 103
92. Associated points 104

XIII. PLANE-COORDINATES

93. Plane-coordinates 106
94. Principle of duality 106
95. Quadric envelope 107
96. Conic 108
97. Cone 110
98. Joachimsthal's quadratic equation 110
99. Range of quadrics 112
100. Confocal quadrics 112

APPENDICES

I. Matrices 115
II. Solutions 117
INDEX 135

CHAPTER I

INTRODUCTION

1. Point coordinates

In a plane the position of a point is determined by its perpendicular distances from two mutually orthogonal straight lines.

Analogously, in space we choose three mutually orthogonal straight lines $X'OX$, $Y'OY$, $Z'OZ$ called the x-axis, y-axis and z-axis respectively (Fig. 1). The convention is made that the directions OX, OY and OZ are positive whilst the opposite directions OX', OY' and OZ' are negative. Then the position of a point P is *uniquely* determined by the perpendicular distances LP, MP and NP to the planes OYZ, OZX and OXY respectively. These distances, which we shall denote by x, y and z respectively, are called the coordinates of the point P with respect to the given axes. The planes OYZ, OZX and OXY are called the **coordinate planes** and O is called the **origin**. We shall write $P \equiv (x, y, z)$ or $P \equiv (x_P, y_P, z_P)$ if we wish to emphasize that x_P, y_P and z_P are the coordinates of a particular point P.

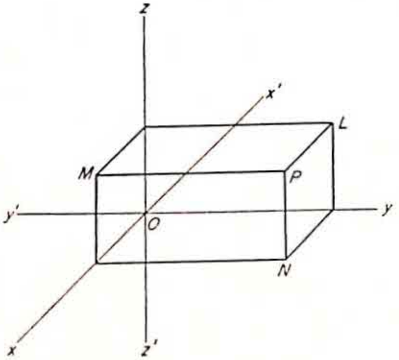

FIG. 1

The coordinate planes divide space into eight octants and the coordinates are signed quantities, their signs depending on the octant of space in which P lies. For example, Q, the mirror image of

P in the plane ZOX, has coordinates $(x_P, -y_P, z_P)$ whilst the mirror image of Q in the YOZ plane has coordinates $(-x_P, -y_P, z_P)$. Further, we have $L \equiv (0, y_P, z_P)$, $M \equiv (x_P, 0, z_P)$, $N \equiv (x_P, y_P, 0)$ and $O \equiv (0, 0, 0)$.

Figure 1 has been drawn so that OX, OY and OZ form a **right-handed** system of orthogonal axes. That is, a right-handed corkscrew rotating from OY to OZ through a right angle advances in the positive direction OX. Similarly, if the corkscrew rotates from OZ to OX or OX to OY through a right angle, it advances in the directions OY and OZ respectively. If we reverse the direction of any one of the axes, we obtain a **left-handed** system, whilst if we reverse the direction of two of the axes simultaneously the system remains right-handed. We shall always employ a right-handed system of axes.

2. Distance between two points

In Fig. 2, draw PL and QM perpendicular to the OXY plane, intersecting it at L and M. Next draw PN in the plane $PQML$ perpendicular to QM. Finally, draw LS and MS in the OXY plane parallel to OY and OX respectively. Then PNQ and LSM are triangles with right angles at N and S respectively. Hence

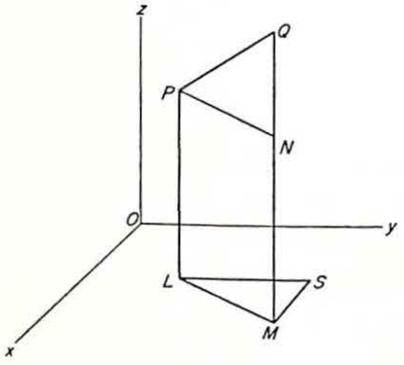

FIG. 2

$$PQ^2 = PN^2 + NQ^2 \text{ and } PN^2 = LM^2 = LS^2 + SM^2.$$

But

$$SM = x_M - x_S = x_Q - x_L = x_Q - x_P;$$

$$LS = y_S - y_L = y_M - y_P = y_Q - y_P$$

and

$$NQ = z_Q - z_N = z_Q - z_P.$$

Thus

$$PQ^2 = (x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2. \qquad (2.1)$$

In particular, the distance of P from the origin is $\sqrt{x_P{}^2 + y_P{}^2 + z_P{}^2}$.

Example 1. Calculate the distances between the following pairs of points: (i) (1, 2, 3), (13, 6, 6); (ii) $(-1, 0, 2)$, $(2, -1, 3)$; (iii) $(t^2, 1, t)$, $(1, -t^2, 3t)$.

3. Joachimsthal's section formula

We often require the coordinates of the point P which divides the line joining A and B in the ratio $AP{:}PB = \sigma{:}\tau$. In Fig. 3, draw AL, BM and PN perpendicular to the OXY plane. Their points of intersection L, M and N with that plane are collinear. Now draw AST in the plane $APBMNL$ parallel to LNM.

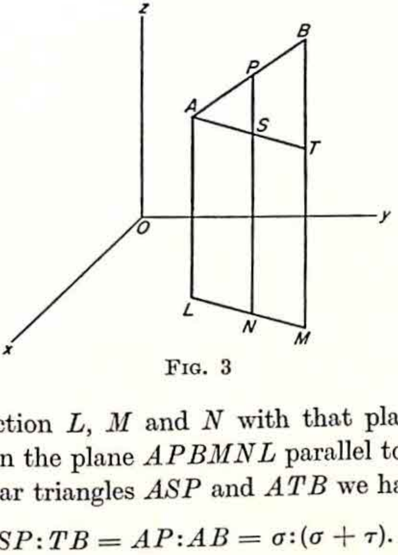

FIG. 3

From the similar triangles ASP and ATB we have

$$SP{:}TB = AP{:}AB = \sigma{:}(\sigma + \tau).$$

That is, $(z_P - z_A){:}(z_B - z_A) = \sigma{:}(\sigma + \tau)$. Likewise, we obtain equations in x and y. Solving these equations we have **Joachimsthal's section formulae**.

$$x_P = \frac{\tau x_A + \sigma x_B}{\tau + \sigma}; \quad y_P = \frac{\tau y_A + \sigma y_B}{\tau + \sigma}; \quad z_P = \frac{\tau z_A + \sigma z_B}{\tau + \sigma}. \qquad (3.1)$$

These formulae also apply when P lies outside AB, in which case the ratio $\sigma{:}\tau$ is negative being numerically greater or less than unity according as B lies between P and A or A lies between P and B respectively. The ratio $\sigma{:}\tau$ never equals -1 for any finite position of P.

If the point Q divides AB in the ratio $-\sigma:\tau$ then we say that P and Q are **harmonic conjugates** with respect to A and B. It follows that the coordinates of Q are given by

$$x_Q = \frac{\tau x_A - \sigma x_B}{\tau - \sigma}; \quad y_Q = \frac{\tau y_A - \sigma y_B}{\tau - \sigma}; \quad z_Q = \frac{\tau z_A - \sigma z_B}{\tau - \sigma}. \qquad (3.2)$$

Example 2. Show that the coordinates of the mid-point of PQ are $(\frac{1}{2}(x_P + x_Q), \frac{1}{2}(y_P + y_Q), \frac{1}{2}(z_P + z_Q))$.

Example 3. In what ratio does the point $(1, 0, -3)$ divide the join of $(-1, -2, 3)$ and $(2, 1, -6)$?

Example 4. P and Q are harmonic conjugates with respect to A and B and $AP:PB = l$. Show that $(1 - l^2)PQ = 2lAB$.

4. Direction-cosines

The **direction-angles** of the directed straight line PQ are defined to be the angles α, β and γ which the line OR, through the origin parallel to PQ and having the same sense (as indicated by the arrows in Fig. 4), makes with the positive directions of the coordinate axes. Draw parallels through P to the axes and drop perpendiculars to them from Q. (In Fig. 4, only the parallel PM to the y-axis is shown.) It follows that

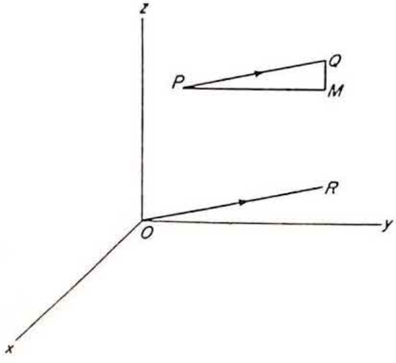

FIG. 4

$$x_Q - x_P = PQ \cos\alpha; \; y_Q - y_P = PQ \cos\beta; \; z_Q - z_P = PQ \cos\gamma.$$

The cosines of the direction-angles are called **direction-cosines.** We introduce for brevity the notation $l \equiv \cos\alpha$, $m \equiv \cos\beta$, $n \equiv \cos\gamma$ and refer to the direction $[l, m, n]$, meaning the direction specified by the direction-cosines l, m and n. The direction-cosines of PQ are thus

$$l = \frac{x_Q - x_P}{PQ}; \quad m = \frac{y_Q - y_P}{PQ}; \quad n = \frac{z_Q - z_P}{PQ}, \tag{4.1}$$

and in virtue of (2.1) they satisfy the equation

$$l^2 + m^2 + n^2 = 1. \tag{4.2}$$

We note that the direction-cosines of QP are $[-l, -m, -n]$.

Example 5. Calculate the direction-cosines of the lines joining the following pairs of points: (i) $(3, -1, 1)$, $(2, 1, -1)$; (ii) $(6, 2, -4)$, $(-1, -2, 1)$.

5. Direction-ratios

Three numbers p, q, r proportional to the direction-cosines l, m, n of a line are called **direction-ratios.** There exist *two* solutions for l, m and n of the equations $l:m:n = p:q:r$ and $l^2 + m^2 + n^2 = 1$, namely,

$$l = \pm \frac{p}{\sqrt{p^2 + q^2 + r^2}}; \quad m = \pm \frac{q}{\sqrt{p^2 + q^2 + r^2}}; \quad n = \pm \frac{r}{\sqrt{p^2 + q^2 + r^2}}, \tag{5.1}$$

where the signs are all positive or all negative. That is, three numbers proportional to the direction-cosines specify the same direction but do not indicate a sense along it. We write for brevity $\{p, q, r\}$ to indicate the triple of direction-ratios p, q and r.

By (4.1) the direction-ratios of the straight line PQ can be taken as the triple $\{x_Q - x_P, y_Q - y_P, z_Q - z_P\}$.

Example 6. Calculate the direction-cosines corresponding to the direction-ratios $\{3, -4, 12\}$.

6. Angle between two straight lines

In space two lines need not lie in the same plane. In that case the lines are said to be skew to one another. The angle between two skew lines is defined to be the angle between the parallels through any point to the given skew lines. Accordingly it suffices to calculate the angle θ between the two straight lines through the origin with direction-cosines $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$. Let P and Q (Fig. 5) be the points along these lines at unit distance from the origin. Then $P \equiv (l_1, m_1, n_1)$ and $Q \equiv (l_2, m_2, n_2)$. Further $PQ^2 = OP^2 + OQ^2 - 2OP \,.\, OQ \cos\theta$. Hence

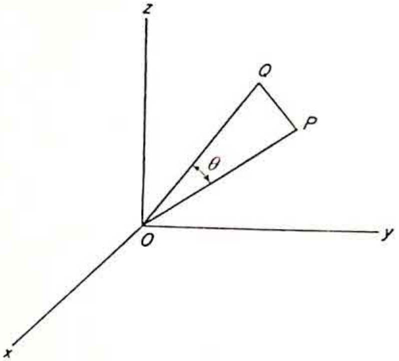

FIG. 5

$$(l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2 = 1 + 1 - 2\cos\theta.$$

In virtue of (4.2) we obtain

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2. \tag{6.1}$$

If the two lines are specified by the direction-ratios $\{p_1, q_1, r_1\}$ and $\{p_2, q_2, r_2\}$, it follows from (5.1) that

$$\cos\theta = \pm \frac{p_1 p_2 + q_1 q_2 + r_1 r_2.}{\sqrt{(p_1{}^2 + q_1{}^2 + r_1{}^2)(p_2{}^2 + q_2{}^2 + r_2{}^2)}}. \tag{6.2}$$

Note that (6.1) yields a unique angle in the range $0 \leq \theta \leq \pi$ for the angle between two directed lines, whilst (6.2) does not differentiate between the acute angle the two lines make with one another and its supplement.

By means of Lagrange's identity

$$(l_1{}^2 + m_1{}^2 + n_1{}^2)(l_2{}^2 + m_2{}^2 + n_2{}^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$
$$\equiv (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2 \tag{6.3}$$

and (4.2) we deduce that

$$\sin^2\theta = (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2 \tag{6.4}$$

with the corresponding result for direction-ratios in the form

$$\sin^2 \theta = \frac{(q_1r_2 - q_2r_1)^2 + (r_1p_2 - r_2p_1)^2 + (p_1q_2 - p_2q_1)^2}{(p_1{}^2 + q_1{}^2 + r_1{}^2)(p_2{}^2 + q_2{}^2 + r_2{}^2)}. \tag{6.5}$$

Hence two straight lines are mutually orthogonal if and only if

$$p_1p_2 + q_1q_2 + r_1r_2 = 0 \tag{6.6}$$

and are parallel to one another if and only if

$$p_1/p_2 = q_1/q_2 = r_1/r_2. \tag{6.7}$$

This latter deduction is obvious by the definition of a direction-ratio.

We often require the direction-ratios $\{\lambda, \mu, \nu\}$ of the line orthogonal to two given lines with the direction-ratios $\{p_1, q_1, r_1\}$ and $\{p_2, q_2, r_2\}$. By (6.6) we have

$$p_1\lambda + q_1\mu + r_1\nu = 0; \quad p_2\lambda + q_2\mu + r_2\nu = 0.$$

Thus

$$\lambda/(q_1r_2 - q_2r_1) = \mu/(r_1p_2 - r_2p_1) = \nu/(p_1q_2 - p_2q_1)$$

and so the required direction may be taken as

$$\{q_1r_2 - q_2r_1, \quad r_1p_2 - r_2p_1, \quad p_1q_2 - p_2q_1\}. \tag{6.8}$$

Example 7. Calculate the angle between the straight lines joining the points $A \equiv (-1, -2, 1)$, $B \equiv (1, -1, -3)$ and the points $C \equiv (0, -1, -2)$, $D \equiv (-1, 1, -1)$. Further, obtain the direction-cosines of the straight line which is perpendicular to these two straight lines.

7. Projections

Let $P_1, P_2, \ldots, P_n$ be n points in space. Then it follows immediately from a diagram that the sum of the projections of P_1P_2, $P_2P_3, \ldots, P_{n-1}P_n$ on any straight line is equal to the projection of P_1P_n on that line.

We wish to calculate the length of the projection of PQ on the line whose direction-ratios are $[l, m, n]$. In Fig. 6 draw PR parallel to the x-axis and QS parallel to the z-axis. RS is the intersection of the plane through PR parallel to OXY and the plane through QS parallel to OYZ and so RS is parallel to the y-axis.

We have $PR = x_Q - x_P$; $RS = y_Q - y_P$ and $SQ = z_Q - z_S$. The projection of PQ is equal to the sum of the projections of PR,

RS and SQ on the line with direction-cosines $[l, m, n]$. Hence the required projection is

$$l(x_Q - x_P) + m(y_Q - y_P) + n(z_Q - z_P). \tag{7.1}$$

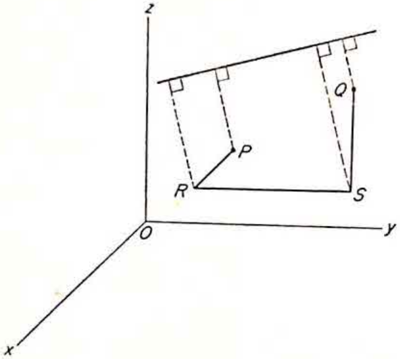

FIG. 6

Example 8. Calculate the projection of AB on the straight line CD and of CD on the straight line AB, where A, B, C and D are the points of Example 7.

Miscellaneous examples

1. Show that the straight lines joining each vertex of a tetrahedron to the centroid of the opposite face and the straight lines joining the mid-points of opposite edges are all concurrent.

2. Show that the straight line which is equally inclined to the three coordinate axes makes an angle $\tan^{-1}\sqrt{2}$ with each axis.

3. Calculate the angle between any two diagonals of a cube.

4. Show that the four points (a, b, c), (b, c, a), (c, a, b) and $(a + b + c, a + b + c, a + b + c)$ form the vertices of a regular tetrahedron if $bc + ca + ab = 0$.

Answers

1. (i) 13; (ii) $\sqrt{11}$; (iii) $\sqrt{2}(t^2 + 1)$. **3.** 2:1. **5.** (i) $[-1/3, 2/3, -2/3]$; (ii) $[-7\sqrt{10}/30; -2\sqrt{10}/15, \sqrt{10}/6]$. **6.** $\pm[3/13, -4/13, 12/13]$. **7.** $\cos^{-1}(2\sqrt{14}/21)$, $\pm[9/\sqrt{110}, 2/\sqrt{110}, 5/\sqrt{110}]$. **8.** $4/\sqrt{6}$, $4/\sqrt{21}$.

Miscellaneous example. 3. $\cos^{-1}(1/3)$.

CHAPTER II

PLANE—STRAIGHT LINE

8. Equation of a plane

Consider the plane ABC (Fig. 7). Let the perpendicular ON from the origin to this plane be of length d and have direction-cosines $[l, m, n]$. The coordinates of N are (dl, dm, dn). Select any point $P \equiv (x, y, z)$ in the plane ABC. Then the direction-ratios of PN are $\{x - dl,\ y - dm,\ z - dn\}$. But PN and ON are mutually orthogonal and so $l(x - dl) + m(y - dm) + n(z - dn) = 0$. In virtue of (4.2) the coordinates of any point on the plane satisfy the linear equation

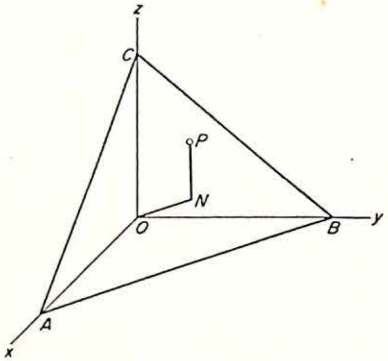

Fig. 7

$$lx + my + nz - d = 0. \tag{8.1}$$

Conversely, consider the general linear equation

$$\lambda x + \mu y + \nu z + \rho = 0, \tag{8.2}$$

where λ, μ and ν are not all zero.

Let A and B be *any* two points on the surface F given by equation (8.2). Then

$$\lambda x_A + \mu y_A + \nu z_A + \rho = 0; \quad \lambda x_B + \mu y_B + \nu z_B + \rho = 0.$$

It immediately follows that

$$\lambda\left(\frac{\tau x_A + \sigma x_B}{\tau + \sigma}\right) + \mu\left(\frac{\tau y_A + \sigma y_B}{\tau + \sigma}\right) + \nu\left(\frac{\tau z_A + \sigma z_B}{\tau + \sigma}\right) + \rho = 0. \quad (8.3)$$

From (3.1) we now see that all points of the straight line AB lie on F. But A and B are arbitrary and so the surface F is such that the line joining any two points on it lies completely in the surface. That is, F is a plane. Consequently, a linear equation always represents a plane.

By comparing (8.1) and (8.2) we see that $\{\lambda, \mu, \nu\}$ are the *direction-ratios* of the normal to the plane.

There are three effective constants, namely the ratios $\lambda:\mu:\nu:\rho$, in equation (8.2). Therefore, a plane is in general determined by three conditions.

For example, the plane which intercepts distances a, b, and c along the respective axes passes through the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Accordingly its equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \quad (8.4)$$

To obtain the equation of the plane PQR, we eliminate λ, μ, ν and ρ from equation (8.2) and the three equations obtained from it by the fact that P, Q and R lie on this plane. The result can be expressed in the determinantal form

$$\begin{vmatrix} x & y & z & 1 \\ x_P & y_P & z_P & 1 \\ x_Q & y_Q & z_Q & 1 \\ x_R & y_R & z_R & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x - x_P & y - y_P & z - z_P \\ x_P - x_Q & y_P - y_Q & z_P - z_Q \\ x_Q - x_R & y_Q - y_R & z_Q - z_R \end{vmatrix} = 0. \quad (8.5)$$

Example 1. **The equation of the plane through A such that its normal has direction-ratios $\{\lambda, \mu, \nu\}$ is $\lambda(x - x_A) + \mu(y - y_A) + \nu(z - z_A) = 0$.**

Example 2. Show that the four points $(1, 2, -1)$, $(2, -1, 3)$, $(-1, 1, 2)$ and $(3, 3, -4)$ are coplanar.

9. Equations of a straight line

Consider the straight line through A (Fig. 8) with direction-ratios $\{l, m, n\}$. If $P \equiv (x, y, z)$ is any point on this line, its direction-ratios are also $\{x - x_A, y - y_A, z - z_A\}$. Consequently the equations of the straight line are

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n}. \tag{9.1}$$

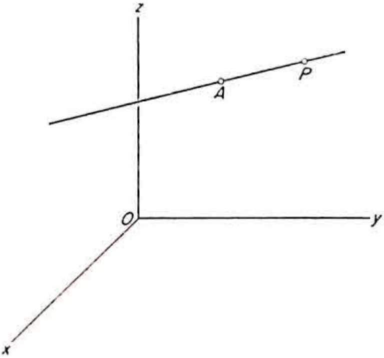

FIG. 8

Setting each of these fractions equal to t, we obtain the parametric equations

$$x = x_A + lt; \quad y = y_A + mt; \quad z = z_A + nt. \tag{9.2}$$

That is, the point $(x_A + lt, y_A + mt, z_A + nt)$ lies on the line for all values of t. Since

$$(x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2 = (l^2 + m^2 + n^2)t^2,$$

it follows that t is the distance between the variable point P and the fixed point A if l, m, n are direction-cosines.

If a straight line is specified by two points A and B, its direction-ratios are $\{x_B - x_A, y_B - y_A, z_B - z_A\}$ and so its equations are

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}, \tag{9.3}$$

or in parametric form

$$x = (1 - t)x_A + tx_B; \quad y = (1 - t)y_A + ty_B; \quad z = (1 - t)z_A + tz_B. \tag{9.4}$$

A straight line is the curve of intersection of two planes. Hence a straight line can also be given by the pair of linear equations

$$\left.\begin{aligned} \lambda_1 x + \mu_1 y + \nu_1 z + \rho_1 = 0, \\ \lambda_2 x + \mu_2 y + \nu_2 z + \rho_2 = 0. \end{aligned}\right\} \tag{9.5}$$

This straight line is orthogonal to the normals to the two planes. These normals have direction-ratios $\{\lambda_1, \mu_1, \nu_1\}$ and $\{\lambda_2, \mu_2, \nu_2\}$. Accordingly, by (6.8) the direction-ratios of the straight line (9.5) are $\{\mu_1\nu_2 - \mu_2\nu_1, \nu_1\lambda_2 - \nu_2\lambda_1, \lambda_1\mu_2 - \lambda_2\mu_1\}$. In order to rewrite the equation of the line in the form (9.1) we need the coordinates of any one point A on it. For example, we may select $z = 0$ and solve (9.5) for the corresponding values of x and y. This selection is always possible if the line is not parallel to the OXY plane. In this exceptional case, take either $x = 0$ or $y = 0$, etc.

Example 3. Verify that the equations (9.5) of a straight line are equivalent to

$$\frac{x - (\rho_2\mu_1 - \rho_1\mu_2)/(\lambda_1\mu_2 - \lambda_2\mu_1)}{\mu_1\nu_2 - \mu_2\nu_1} = \frac{y - (\rho_1\lambda_2 - \rho_2\lambda_1)/(\lambda_1\mu_2 - \lambda_2\mu_1)}{\nu_1\lambda_2 - \nu_2\lambda_1}$$

$$= \frac{z}{\lambda_1\mu_2 - \lambda_2\mu_1}.$$

Example 4. Prove that the straight lines $x - 4y + 7 = 11y + z + 2 = 0$ and $x - 8y - 5 = 3y - z + 1 = 0$ are orthogonal.

10. Straight line and plane

The value of t which corresponds to the intersection of the straight line

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n} = t \tag{10.1}$$

and the plane

$$\lambda x + \mu y + \nu z + \rho = 0 \tag{10.2}$$

satisfies the equation

$$\lambda(x_A + lt) + \mu(y_A + mt) + \nu(z_A + nt) + \rho = 0.$$

That is,

$$t(\lambda l + \mu m + \nu n) + \lambda x_A + \mu y_A + \nu z_A + \rho = 0. \tag{10.3}$$

If $\lambda l + \mu m + \nu n = 0$, then the direction $\{\lambda, \mu, \nu\}$ is perpendicular to the direction $\{l, m, n\}$. Thus the straight line (10.1) is perpendicular to the normal to the plane (10.2) and so the straight line (10.1) is parallel to the plane (10.2). In this case equation (10.3) has no finite root.

If, in addition, $\lambda x_A + \mu y_A + \nu z_A + \rho = 0$ the point lies in the plane (10.2). Accordingly the line (10.1) lies completely in the plane (10.2). In this case equation (10.3) reduces to an identity.

The angle between a straight line and a plane is defined to be the complement of the angle between the straight line and the normal to the plane. Thus the angle θ between the straight line (10.1) and the plane (10.2) is given by

$$\sin\theta = (\lambda l + \mu m + \nu n)/\sqrt{(l^2 + m^2 + n^2)(\lambda^2 + \mu^2 + \nu^2)}. \quad (10.4)$$

Consider the determinantal equation

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0. \quad (10.5)$$

It is linear in x, y and z and so represents a plane. The point A clearly lies on this plane and further the straight line

$$(x - x_A)/l_1 = (y - y_A)/m_1 = (z - z_A)/n_1$$

lies in the plane because the substitutions

$$x - x_A = l_1 t;\ \ y - y_A = m_1 t;\ \ z - z_A = n_1 t$$

make the determinant identically zero. Similarly the line

$$(x - x_A)/l_2 = (y - y_A)/m_2 = (z - z_A)/n_2$$

also lies on the plane. Hence (10.5) is the equation of the plane which passes through A and contains lines through A with the directions $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$.

Example 5. Show that the plane through AB parallel to the straight line $(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n$ is given by

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ l & m & n \end{vmatrix} = 0.$$

Example 6. Obtain the point of intersection of the plane $2x + 3y - 4z + 5 = 0$ and the straight line $(x - 1)/2 = (y - 2)/3 = (z - 3)/4$.

11. Sign of the expression $u = \lambda x + \mu y + \nu z + \rho$

Let the straight line AB cut the plane $u = 0$, at P, where $AP{:}PB = \sigma{:}\tau$. Then the coordinates of P are given by Joachimsthal's section formula (3.1). Since P lies on $u = 0$ equation (8.3) is satisfied. Hence

$$\frac{\sigma}{\tau} = -\frac{\lambda x_A + \mu y_A + \nu z_A + \rho}{\lambda x_B + \mu y_B + \nu z_B + \rho} = -\frac{u_A}{u_B}\ \text{(say)}.$$

If u_A and u_B have the same sign, the ratio $\sigma:\tau$ is negative and so P lies outside AB. Accordingly A and B lie on the same side of the plane $u = 0$. If, on the other hand, u_A and u_B have opposite signs, the ratio $\sigma:\tau$ is positive and so P lies between A and B. In this case A and B lie on opposite sides of the plane $u = 0$.

Thus the plane $u = 0$ divides space into two regions, such that $u > 0$ in one of them and $u < 0$ in the other.

Example 7. If $\rho > 0$, show that the origin and the points (λ, μ, ν), $(\mu - \nu, \nu - \lambda, \lambda - \mu)$ all lie on the same side of the plane $\lambda x + \mu y + \nu z + \rho = 0$.

12. Distance from a point to a plane

We wish to calculate the perpendicular distance d from the point P to the plane $u \equiv \lambda x + \mu y + \nu z + \rho = 0$. The straight line through P perpendicular to this plane is

$$(x - x_P)/\lambda = (y - y_P)/\mu = (z - z_P)/\nu.$$

Hence the foot N of the perpendicular from P to this plane has the coordinates $(x_P + \lambda t, y_P + \mu t, z_P + \nu t)$ for some value of t. Since N lies in the plane $u = 0$, we have

$$\lambda(x_P + \lambda t) + \mu(y_P + \mu t) + \nu(z_P + \nu t) + \rho = 0$$

and so

$$t = -(\lambda x_P + \mu y_P + \nu z_P + \rho)/(\lambda^2 + \mu^2 + \nu^2).$$

Further, $PN^2 = (\lambda^2 + \mu^2 + \nu^2)t^2$ and we obtain

$$d = PN = \pm \frac{\lambda x_P + \mu y_P + \nu z_P + \rho}{\sqrt{\lambda^2 + \mu^2 + \nu^2}}. \tag{12.1}$$

We make the convention that the sign be taken in such a way that the perpendicular from the origin is positive. By the previous section it follows that d is positive for all points on the same side of the plane as the origin, and negative for all points on the other side of the plane.

Example 8. Find the locus of a point the sum of the squares of whose distances from the planes $-x + y + z = 0$, $x - y + z = 0$, $x + y - z = 0$ is three times the square of its distance from the origin.

Example 9. Find two points on the straight line $\frac{1}{3}(x + 2) = -(y + 3) = \frac{1}{2}(z - 2)$ which are at unit distance from the plane $x + 2y - 2z + 3 = 0$.

13. Pair of planes

From the equations

$$u_1 \equiv \lambda_1 x + \mu_1 y + \nu_1 z + \rho_1 = 0,$$
$$u_2 \equiv \lambda_2 x + \mu_2 y + \nu_2 z + \rho_2 = 0,$$

of two planes we may set up the equation

$$\sigma u_1 + \tau u_2 = 0. \tag{13.1}$$

This equation is linear in x, y and z and is satisfied by the solution of the equations $u_1 = u_2 = 0$. Accordingly (13.1) represents a plane passing through the line $u_1 = u_2 = 0$. As the ratio $\sigma : \tau$ varies, equation (13.1) represents all the planes through this common straight line, called an axis. We also say that the planes form a pencil of planes.

If the planes $u_1 = 0$ and $u_2 = 0$ are parallel, we have $\lambda_1/\lambda_2 = \mu_1/\mu_2 = \nu_1/\nu_2$ and so $\sigma u_1 + \tau u_2 = 0$ represents all the planes parallel to $u_1 = 0$ or $u_2 = 0$.

The angle between two planes is defined to be the angle between their normals. Consequently the angle θ between the two planes $u_1 = 0$ and $u_2 = 0$ is given by

$$\cos\theta = (\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2)/\sqrt{(\lambda_1^2 + \mu_1^2 + \nu_1^2)(\lambda_2^2 + \mu_2^2 + \nu_2^2)}. \tag{13.2}$$

Accordingly the two planes are orthogonal if

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0. \tag{13.3}$$

Example 10. Find the equation of the plane through the straight line $(x - 1)/2 = (y + 1)/-1 = (z - 3)/3$, (i) perpendicular to the plane $8x - 2y + 9z + 6 = 0$; (ii) parallel to the straight line $x = y = z$.

Example 11. Obtain the condition that the three planes $\lambda x + \mu y + \nu z = 0$, $\mu x + \nu y + \lambda z = 0$ and $\nu x + \lambda y + \mu z = 0$ have a straight line in common.

14. Bundle of planes

Consider three distinct planes $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$ and set up the equation

$$\sigma u_1 + \tau u_2 + \kappa u_3 = 0. \tag{14.1}$$

This equation is linear in x, y and z and is satisfied by the solution of the equations $u_1 = u_2 = u_3 = 0$. Thus (14.1) represents a plane passing through the point of intersection of $u_1 = 0$, $u_2 = 0$ and

$u_3 = 0$. As the ratios $\sigma : \tau : \kappa$ vary, equation (14.1) represents all the planes passing through this point of intersection, called the **vertex.** We say that the planes form a **bundle** of planes.

A more detailed discussion of all the special cases which may occur is given later in §55.

Example 12. Show that the four planes $ay + bz + c = 0$, $ax - dz - e = 0$, $bx + dy - f = 0$ and $cx + ey + fz = 0$ have a straight line in common if $af - be + cd = 0$.

Example 13. Show that the four planes $x + 2y - 4z + 9 = 0$, $3x - y + z - 6 = 0$, $2x + y + z - 3 = 0$ and $7x - 2y - 4z - 1 = 0$ have a common point.

15. Shortest distance between two skew lines

The shortest distance between a point and a straight line is the perpendicular to it from the point. Consequently the shortest distance between two skew lines is along the line which is perpendicular to both of them.

Let the straight lines be

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n} \tag{15.1}$$

and

$$\frac{x - x_B}{p} = \frac{y - y_B}{q} = \frac{z - z_B}{r}. \tag{15.2}$$

By (6.8) the direction of the common perpendicular is $\{mr - nq, np - lr, lq - mp\}$. The corresponding direction-cosines are

$$[(mr - nq)/k, \quad (np - lr)/k, \quad (lq - mp)/k]$$

where

$$k^2 = (mr - nq)^2 + (np - lr)^2 + (lq - mp)^2.$$

The shortest distance d is the projection (Fig. 9) of AB on the common perpendicular. From (7.1) we now obtain

$$d = \{(x_A - x_B)(mr - nq) + (y_A - y_B)(np - lr) + (z_A - z_B)(lq - mp)\}/k,$$

or in determinantal form

$$d = \begin{vmatrix} x_A - x_B & y_A - y_B & z_A - z_B \\ l & m & n \\ p & q & r \end{vmatrix} \div \sqrt{\Sigma(mr - nq)^2}. \tag{15.3}$$

Let the shortest distance intersect the straight lines (15.1) and (15.2) at P and Q respectively. Then the shortest distance PQ is the line of intersection of the planes APQ and BPQ. In virtue of (10.5) the equations of the shortest distance are

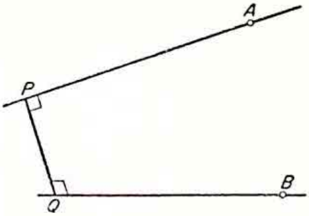

FIG. 9

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ l & m & n \\ mr - nq & np - lr & lq - mp \end{vmatrix} = \begin{vmatrix} x - x_B & y - y_B & z - z_B \\ p & q & r \\ mr - nq & np - lr & lq - mp \end{vmatrix} = 0.$$

Example 14. Calculate the length and obtain the equations of the common perpendicular to the two straight lines. (i) $(x - 4)/2 = (y - 5)/3 = (z - 6)/4$ and $x + 1 = y + 2 = (z + 3)/3$; (ii) $3x + y + z - 13 = x + y - z - 1 = 0$ and $x + y + z = 3x + y + 6z - 19 = 0$.

16. Coplanar lines

If the two straight lines (15.1) and (15.2) are coplanar, they are either parallel or intersect. In the former case $l/p = m/q = n/r$ whilst in the latter case the shortest distance between them is zero. Hence by (15.3)

$$\begin{vmatrix} x_A - x_B & y_A - y_B & z_A - z_B \\ l & m & n \\ p & q & r \end{vmatrix} = 0. \tag{16.1}$$

Note that this determinant also vanishes when the lines are parallel and so (16.1) is the necessary and sufficient condition that the two lines are coplanar.

Example 15. Show that the straight line through (α, β, γ) with direction-ratios $\{l, m, n\}$ is coplanar with the straight line through (l, m, n) with direction-ratios $\{\alpha, \beta, \gamma\}$.

17. Rectangular axes related to two skew lines

Let $PQ = 2c$ be the shortest distance between the two skew lines PA and QB (Fig. 10). Select the origin O at the mid-point of PQ and draw parallels OA' and OB' to PA and QB respectively. Select the bisectors of the angles between OA' and OB' as the x-axis and y-axis according to the figure. Finally select the z-axis along OP. We have $P \equiv (0, 0, c)$ and $Q \equiv (0, 0, -c)$. The direction-cosines of PA are $[-\sin\alpha, \cos\alpha, 0]$ and of QB are $[\sin\alpha, \cos\alpha, 0]$ where 2α is the angle between PA and QB. Hence the equations of PA and QB are

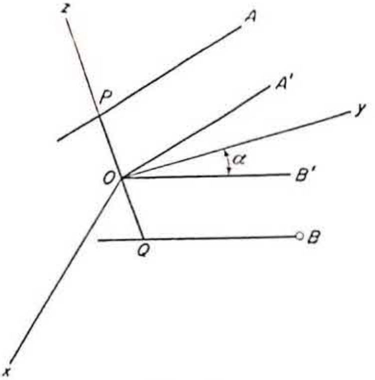

FIG. 10

$$\frac{x}{-\sin\alpha} = \frac{y}{\cos\alpha} = \frac{z-c}{0} \quad \text{and} \quad \frac{x}{\sin\alpha} = \frac{y}{\cos\alpha} = \frac{z+c}{0}. \qquad (17.1)$$

Putting $k = \cot\alpha$, we may write these equations

$$y = -kx, \quad z = c \quad \text{and} \quad y = kx, \quad z = -c. \qquad (17.2)$$

Example 16. A and B are fixed points of two skew lines AL and BM. R and S lie on AL and BM respectively. Obtain the locus of the mid-point of RS if (i) $AR = BS$; (ii) $AR + BS$ is constant; (iii) $AR^2 + BS^2$ is constant; (iv) $AR \,.\, BS$ is constant.

Miscellaneous examples

1. Show that the point in which the line AB is intersected by the perpendicular to it from the origin is $([x_A(y_B{}^2 + z_B{}^2) + x_B(y_A{}^2 + z_A{}^2) - (x_A + x_B)(y_A y_B + z_A z_B)]/AB^2$, etc.).

2. Prove that the distance p of the point P from the straight line AB is given by

$$p^2 = \Sigma\{(y_A - y_B)(z_P - z_A) - (z_A - z_B)(y_P - y_A)\}^2/AB^2.$$

3. Show that the mirror image of the point A in the plane $\lambda x + \mu y + \nu z + \rho = 0$ has coordinates $([(\mu^2 + \nu^2 - \lambda^2)x_A - 2\lambda\mu y_A - 2\lambda\nu z_A - 2\lambda\rho]/(\lambda^2 + \mu^2 + \nu^2)$, etc.).

4. Outline a method to obtain the equations of the mirror image of a given straight line in a given plane.

5. Show that the equations of the orthogonal projection of the straight line $(x - x_A)/l = (y - y_A)/m = (z - z_A)/n$ on the plane $u \equiv \lambda x + \mu y + \nu z + \rho = 0$ are

$$u = \left(\frac{\mu}{m} - \frac{\nu}{n}\right)\frac{x - x_A}{l} + \left(\frac{\nu}{n} - \frac{\lambda}{l}\right)\frac{y - y_A}{m} + \left(\frac{\lambda}{l} - \frac{\mu}{m}\right)\frac{z - z_A}{n} = 0.$$

6. If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is the equation of two planes intersecting at an angle $\pi/3$, prove that

$$4(f^2 + g^2 + h^2) = 3(a^2 + b^2 + c^2) + 10(bc + ca + ab).$$

7. If the perpendiculars from a variable point on two fixed non-intersecting straight lines are orthogonal to one another, prove that the locus of the point is represented by an equation of the second degree.

8. Prove that the equation

$$a(z - x)(x - y) + b(x - y)(y - z) + c(y - z)(z - x) = 0$$

represents two planes whose line of intersection is equally inclined to the three coordinate axes.

9. Prove that the volume of the tetrahedron joining the origin to the points A, B and C is

$$\tfrac{1}{6}\begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix}.$$

Answers

6. (5/3, 3, 13/3). **8.** $3(x^2 + y^2 + z^2) + yz + zx + xy = 0$. **9.** $(-14, 1, -6)$ and $(-8, -1, -2)$. **10.** (i) $3x - 6y - 4z + 3 = 0$; (ii) $4x - y - 3z + 4 = 0$. **11.** $\lambda^3 + \mu^3 + \nu^3 - 3\lambda\mu\nu = 0$. **13.** $(1, -1, 2)$. **14.** (i) $\sqrt{30}/15$, $5x + 22y - 19z - 16 = 5x + 16y - 7z + 16 = 0$; (ii) $17/\sqrt{59}$, $17x + 8y + z - 62 = 27x + 37y + 122 + 95 = 0$. **16.** (i) straight line; (ii) straight line; (iii) ellipse; (iv) hyperbola (see worked out solutions, p. 117).

CHAPTER III

SPHERE

18. Equation of sphere

If $P \equiv (x, y, z)$ is any point on the sphere (Fig. 11) with centre at C and radius r, we have

$$(x - x_C)^2 + (y - y_C)^2 + (z - z_C)^2 = r^2, \qquad (18.1)$$

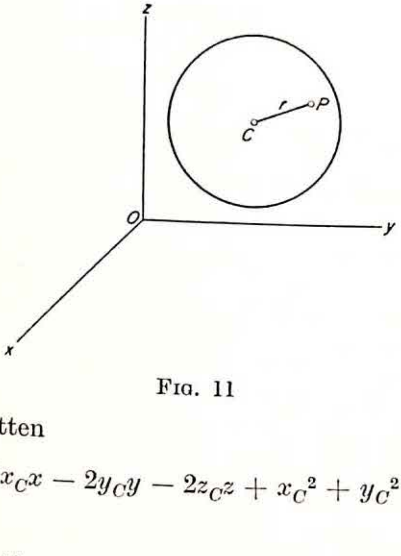

Fig. 11

which can be written

$$x^2 + y^2 + z^2 - 2x_C x - 2y_C y - 2z_C z + x_C{}^2 + y_C{}^2 + z_C{}^2 - r^2 = 0.$$

We see that:

(1) this equation is quadratic in x, y and z;
(2) the coefficients of x^2, y^2 and z^2 are equal;
(3) the coefficients of yz, zx and xy are zero.

Conversely, the most general equation satisfying these three conditions is

$$ax^2 + ay^2 + az^2 + 2px + 2qy + 2rz + d = 0. \quad (a \neq 0). \qquad (18.2)$$

We may write this equation in the form

$$(x + p/a)^2 + (y + q/a)^2 + (z + r/a)^2 = (p^2 + q^2 + r^2 - ad)/a^2,$$

which shows by comparison with (18.1) that it represents a sphere with centre at the point $(-p/a,\ -q/a,\ -r/a)$ and radius $\sqrt{p^2 + q^2 + r^2 - ad}/a$. The centre of this sphere is always a real point (a, p, q, r and d being real), but its radius is real if and only if $p^2 + q^2 + r^2 > ad$. If $p^2 + q^2 + r^2 = ad$, then there is only one real point $(-p/a, -q/a, -r/a)$ on the sphere, and we refer to it as a **point-sphere.** If $p^2 + q^2 + r^2 < ad$ there are no real points on the sphere (18.2) and we call it a **virtual** sphere.

There is no loss in generality if we take $a = 1$, and so the standard form of the equation of a sphere will be

$$F \equiv x^2 + y^2 + z^2 + 2px + 2qy + 2rz + d = 0. \tag{18.3}$$

The radius of this sphere is $\sqrt{p^2 + q^2 + r^2 - d}$ and its centre is at $(-p, -q, -r)$.

In a problem, if possible, select the origin as centre and the simpler equation

$$x^2 + y^2 + z^2 - r^2 = 0 \tag{18.4}$$

now represents a sphere with radius r.

Example 1. Find the equation of the sphere with centre at $(-2, 3, -1)$ and whose radius is 3.

Example 2. Obtain the coordinates of the centre and the radius of the sphere represented by $6x^2 + 6y^2 + 6z^2 + 10x - 8y + 2z + 1 = 0$.

Example 3. Obtain the equation of the sphere through the four points $(1, -1, 0)$, $(2, 3, -1)$, $(0, 1, -2)$ and $(-1, 2, 1)$.

Example 4. Show that the equation of the sphere having AB as a diameter is $(x - x_A)(x - x_B) + (y - y_A)(y - y_B) + (z - z_A)(z - z_B) = 0$.

Example 5. If $F(x, y, z) \equiv x^2 + y^2 + z^2 + 2px + 2qy + 2rz + d$, show that $F(x_P, y_P, z_P)$ is the square of the length of the tangent from P to the sphere $F = 0$. Hence the point P lies inside or outside the sphere according as $F(x_P, y_P, z_P)$ is less than or greater than zero respectively.

Example 6. Show that the equation $a(x^2 + y^2 + z^2) + 2px + 2qy + 2rz + d + k(\lambda x + \mu y + \nu z + \rho) = 0$ represents a sphere through the circle $a(x^2 + y^2 + z^2) + 2px + 2qy + 2rz + d = \lambda x + \mu y + \nu z + \rho = 0$.

19. Conjugate points

Consider the two points A and B which do not lie on the sphere

$$F(x, y, z) \equiv F \equiv x^2 + y^2 + z^2 + 2px + 2qy + 2rz + d = 0. \tag{19.1}$$

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Then the coordinates of the point P which divides AB in the ratio $\sigma:\tau$ are given by (3.1). If this point lies on the sphere, we have

$$F\left(\frac{\tau x_A + \sigma x_B}{\tau + \sigma}, \frac{\tau y_A + \sigma y_B}{\tau + \sigma}, \frac{\tau z_A + \sigma z_B}{\tau + \sigma}\right) = 0.$$

On multiplication by $(\tau + \sigma)^2$, this equation simplifies to

$$\tau^2 F_A + 2\tau\sigma T_{AB} + \sigma^2 F_B = 0, \tag{19.2}$$

where

$$F_A \equiv F(x_A, y_A, z_A); \quad F_B \equiv F(x_B, y_B, z_B)$$

$$T_{AB} \equiv T_{BA} \equiv x_A x_B + y_A y_B + z_A z_B + p(x_A + x_B) + q(y_A + y_B) + r(z_A + z_B) + r.$$

Equation (19.2) is called Joachimsthal's quadratic equation and its roots correspond to the two points of intersection P and Q (Fig. 12) of AB and the sphere.

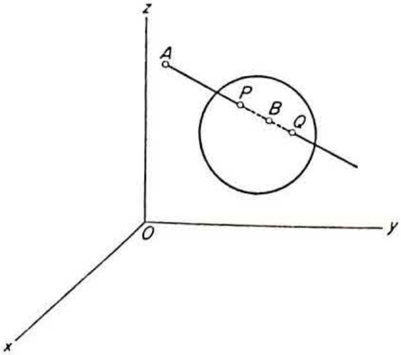

Fig. 12

Two points A and B are said to be **conjugate** with respect to the sphere if the points of intersection P and Q divide A and B harmonically. In this case the roots of the quadratic equation (19.2) in σ/τ are equal in magnitude but opposite in sign. That is, the sum of the roots of (19.2) is zero. Accordingly, the necessary and sufficient condition that the points A and B be conjugate with respect to the sphere (19.1) is $T_{AB} = 0$.

Example 7. For what values of λ are the points $(-\lambda, 2, \lambda)$, $(\lambda, 2, 3)$ conjugate with respect to the sphere $x^2 + y^2 + z^2 - 6x + 2y - 4z + 4 = 0$.

20. Pole and polar

The **polar** of a point A with respect to a sphere is defined to be the locus of all points conjugate to A. The point B is conjugate to A with respect to the sphere (19.1) if $T_{AB} = 0$. The locus of B is obtained by allowing the coordinates x_B, y_B, z_B to become variable; this is achieved by omitting the suffix B. We then find that the locus is given by

$$T_A \equiv (x_A + p)x + (y_A + q)y + (z_A + r)z + px_A + qy_A + rz_A + d = 0. \qquad (20.1)$$

This equation is linear in x, y and z and so the polar of a point A is a plane which is perpendicular to the join of A and the centre of the sphere.

Conversely, the **pole** of a plane is defined to be that point whose polar is the given plane. Let us now obtain the coordinates of the pole A of the plane $\lambda x + \mu y + \nu z + \rho = 0$. The polar of A is given by (20.1), which must then represent the same plane as $\lambda x + \mu y + \nu z + \rho = 0$. The comparison of coefficients yields

$$x_A + p = \kappa\lambda; \quad y_A + q = \kappa\mu; \quad z_A + r = \kappa\nu; \quad px_A + qy_A + rz_A + d = \kappa\rho;$$

where κ is some factor of proportionality. The solution for x_A, y_A and z_A may be found but the formulae are too cumbersome to be quoted. The reader is advised to use the method of this section in relevant examples. (However, compare §62.)

We note that if the sphere is $x^2 + y^2 + z^2 = r^2$, then the two points A and B are conjugate if $x_Ax_B + y_Ay_B + z_Az_B = r^2$, the polar of A is the plane $x_Ax + y_Ay + z_Az - r^2 = 0$ and the pole of $\lambda x + \mu y + \nu z + \rho = 0$ is at the point $(-\lambda r^2/\rho, -\mu r^2/\rho, -\nu r^2/\rho)$.

Example 8. Write down the equations of the polars of the points $(1, 2, -1)$, $(3, 5, -2)$ and $(0, \frac{1}{2}, -\frac{1}{2})$ with respect to the sphere $x^2 + y^2 + z^2 - 3x + y + z - 5 = 0$ and show that they form a pencil of planes.

Example 9. Obtain the coordinates of the poles of the planes $2x + 2y - 3z + 7 = 0$, $4x + z - 3 = 0$ and $5x - y + 3z - 8 = 0$ with respect to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 2z = 0$, and show that they are collinear.

21. Tangent plane and normal line

Let us return to Joachimsthal's equation (19.2), whose roots correspond to the points of intersection of AB and the sphere

(19.1). Suppose A now lies on the sphere (Fig. 13); then $F_A = 0$. The line AB now cuts the sphere in coincident points at A if (19.2)

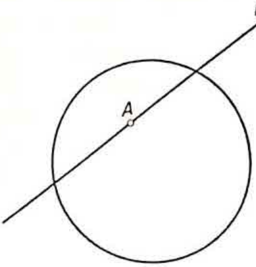

FIG. 13

has coincident roots $\sigma/\tau = 0$. Therefore $T_{AB} = 0$ if B lies on a tangent line at A. Hence the locus of B such that AB is a tangent line is the plane $T_A = 0$, called the **tangent** plane. It is identical in form with (20.1) which gives the polar of A and so the pole of a tangent plane is at its point of contact.

We now ask: "What is the necessary and sufficient condition that the plane $\lambda x + \mu y + \nu z + \rho = 0$ be a tangent plane?" In this case the perpendicular distance from the centre $(-p, -q, -r)$ is equal to the radius $\sqrt{p^2 + q^2 + r^2 - d}$ of the sphere. Thus the necessary and sufficient condition that $\lambda x + \mu y + \nu z + \rho = 0$ be a tangent plane is

$$(\lambda p + \mu q + \nu r - \rho)^2 = (\lambda^2 + \mu^2 + \nu^2)(p^2 + q^2 + r^2 - d). \quad (21.1)$$

If the sphere is $x^2 + y^2 + z^2 - r^2 = 0$, the tangency condition (21.1) simplifies to $r^2(\lambda^2 + \mu^2 + \nu^2) = \rho^2$.

The **normal** is the straight line perpendicular to a tangent plane at its point of contact. Consequently the equations of the normal at A are

$$\frac{x - x_A}{x_A + p} = \frac{y - y_A}{y_A + q} = \frac{z - z_A}{z_A + r}.$$

We readily verify that the centre $(-p, -q, -r)$ lies on every normal.

Example 10. Show that the plane $z = lx + my + r\sqrt{l^2 + m^2 + 1}$ touches the sphere $x^2 + y^2 + z^2 = r^2$ for all values of l and m.

Example 11. Obtain the equation of the spheres which pass through the circle $x^2 + y^2 + z^2 - 13 = x + y + z - 1 = 0$ and touch the plane $2x + 3y - z + 10 = 0$.

22. Angle of intersection of two spheres

The angle of intersection of two spheres is, by definition, the angle between the two tangent planes at a point of intersection. It is a geometrical fact that this angle is the same at all points of the circle of intersection. If this angle is a right angle the two spheres are said to cut orthogonally. Then at a point of intersection the tangent plane to either sphere passes through the centre of the other. It follows that the square of the distance between their two centres is equal to the sum of the squares of their two radii. Let us write

$$F_i \equiv x^2 + y^2 + z^2 + 2p_i x + 2q_i y + 2r_i z + d_i. \qquad (22.1)$$

Then the two spheres $F_1 = 0$ and $F_2 = 0$ cut orthogonally if

$$\begin{aligned}(p_1 - p_2)^2 + (q_1 - q_2)^2 + (r_1 - r_2)^2 \\ = (p_1{}^2 + q_1{}^2 + r_1{}^2 - d_1) + (p_2{}^2 + q_2{}^2 + r_2{}^2 - d_2).\end{aligned}$$

That is,

$$2p_1p_2 + 2q_1q_2 + 2r_1r_2 - d_1 - d_2 = 0. \qquad (22.2)$$

Example 12. Calculate the angle of intersection of the two spheres

$$x^2 + y^2 + z^2 - x + 3y + 2z - 1 = 0$$

and

$$x^2 + y^2 + z^2 + 5x - 2y - z - 9 = 0.$$

23. Radical plane

The square of the length of the tangent from $P \equiv (x, y, z)$ to a sphere is equal to the square of the distance from P to the centre of the sphere minus the square of the radius of the sphere. That is, the square of the length of the tangent from P to the sphere $F_i = 0$ is $(x + p_i)^2 + (y + q_i)^2 + (z + r_i)^2 - (p_i{}^2 + q_i{}^2 + r_i{}^2 - d_i) \equiv F_i$. Accordingly, the locus of points $P \equiv (x, y, z)$ from which the tangents to the spheres $F_1 = 0$ and $F_2 = 0$ are equal is given by

$$\begin{aligned}F_1 - F_2 \equiv 2(p_1 - p_2)x + 2(q_1 - q_2)y \\ + 2(r_1 - r_2)z + d_1 - d_2 = 0. \qquad (23.1)\end{aligned}$$

This equation is linear and so represents a plane, called the **radical plane** of the two spheres. The direction-ratios of the normal to the radical planes are $\{p_1 - p_2, q_1 - q_2, r_1 - r_2\}$, and so the radical plane is orthogonal to the line joining the centres of the two spheres. The radical plane obviously passes through any common points the two spheres may have.

Let us introduce a third sphere $F_3 = 0$. Now we have three radical planes $F_2 = F_3$, $F_3 = F_1$ and $F_1 = F_2$. These three planes form a pencil whose axis is the straight line $F_1 = F_2 = F_3$. This line is called the **radical line** of the three spheres.

Now add a fourth sphere $F_4 = 0$, and we have four radical lines

$$F_2 = F_3 = F_4, \quad F_3 = F_4 = F_1, \quad F_4 = F_1 = F_2 \text{ and}$$
$$F_1 = F_2 = F_3.$$

These four lines are clearly concurrent at the **radical centre** given by the three linear equations $F_1 = F_2 = F_3 = F_4$.

Example 13. Show that the locus of a point from which equal tangents may be drawn to the three spheres $x^2 + y^2 + z^2 - x + y - 3z + 1 = 0$, $x^2 + y^2 + z^2 + 3x - y - z + 2 = 0$ and $x^2 + y^2 + z^2 + x + 3y + z + 3 = 0$ is the straight line $2x + 1 = 2y + 1 = -2z$.

24. Coaxal spheres

Consider the two spheres $F_1 = 0$, $F_2 = 0$ and set up the equation

$$F_1 + kF_2 = 0. \tag{24.1}$$

Except for $k = -1$, which corresponds to the radical plane of $F_1 = 0$ and $F_2 = 0$, equation (24.1) represents a system of spheres. The radical plane of the two distinct spheres corresponding to $k = k_1$ and $k = k_2$ $(k_1 \neq k_2)$ is

$$(F_1 + k_1F_2)/(1 + k_1) - (F_1 + k_2F_2)/(1 + k_2) = 0.$$

This equation reduces to $F_1 - F_2 = 0$ and so (24.1) represents a system of spheres, called a **coaxal** system, with a common radical plane. The centre of the sphere (24.1) is at the point

$$\big(-(p_1 + kp_2)/(1 + k), \quad -(q_1 + kq_2)/(1 + k), \quad -(r_1 + kr_2)/(1 + k)\big). \tag{24.2}$$

Therefore, by (3.1), the centres of all such spheres of a coaxal system are collinear on a line perpendicular to the common radical plane.

Let us choose the common radical plane as the OYZ plane and the line of centres orthogonal to it as the OX axis. From (23.1) we have $q_1 = q_2$, $r_1 = r_2$ and $d_1 = d_2$. The point (24.2) now lies on the x-axis and so $q_1 = q_2 = 0$ and $r_1 = r_2 = 0$. Accordingly we may now write (24.1) on division by $1 + k$ in the standard form

$$x^2 + y^2 + z^2 + 2\sigma x + d = 0, \tag{24.3}$$

where we have put $\sigma = (p_1 + kp_2)/(1 + k)$ and $d_1 = d_2 = d$.

There are three cases,† according as (i) $d < 0$ when all the spheres of the coaxal system have a real circle in common, (ii) $d = 0$ when all the spheres touch each other at the origin and (iii) $d > 0$ when no two spheres of the coaxal system intersect. In this latter case the system contains two point spheres, called **limiting points**.

We now determine all the spheres which intersect *all* spheres of the coaxal system (24.3) orthogonally. Let one such sphere be

$$x^2 + y^2 + z^2 + 2\alpha x + 2\beta y + 2\gamma z + \delta = 0.$$

In virtue of (22.2) we have $\sigma\alpha - d - \delta \equiv 0$ for *all* values of σ and so $\alpha = 0$, $\delta = -d$. Thus *all* spheres of the system

$$x^2 + y^2 + z^2 + 2\beta y + 2\gamma z - d = 0 \tag{24.4}$$

intersect all spheres of the coaxal system (24.3) orthogonally. Note carefully that the spheres (24.4) do *not* form a coaxal system.

Example 14. Obtain the limiting points of the coaxal system of spheres determined by the spheres $2(x^2 + y^2 + z^2) - 6x + 2y + 2z + 7 = 0$ and $4(x^2 + y^2 + z^2) - 14x + 2y + 6z + 17 = 0$.

Miscellaneous examples

1. Obtain the condition that the sphere $x^2 + y^2 + z^2 + 2p_1x + 2q_1y + 2r_1z + d_1 = 0$ should cut the sphere $x^2 + y^2 + z^2 + 2p_2x + 2q_2y + 2r_2z + d_2 = 0$ along a great circle of the latter. Deduce that the spheres which cut two given spheres along great circles, all pass through two fixed points.

2. Find the condition that the two circles $x^2 + y^2 + z^2 + 2p_1x + 2q_1y + 2r_1z + d_1 = \lambda_1x + \mu_1y + \nu_1z + \rho_1 = 0$ and $x^2 + y^2 + z^2 + 2p_2x + 2q_2y + 2r_2z + d_2 = \lambda_2x + \mu_2y + \nu_2z + \rho_2 = 0$ should lie on the same sphere.

3. PQ is the shortest distance between the two straight lines AP and BQ. If AB is of constant length as A and B vary, show that the radius of the sphere $ABPQ$ is constant.

4. Find the condition that the straight line $(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n$ should touch the sphere $x^2 + y^2 + z^2 + 2px + 2qy + 2rz + d = 0$. Hence find the locus of the centres of spheres of constant radius which pass through a given point and touch a given straight line.

5. Obtain the equation of the sphere orthogonal to the four spheres

$$x^2 + y^2 + z^2 + 2p_ix + 2q_iy + 2r_iz + d_i = 0. \quad (i = 1, 2, 3, 4).$$

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† The reader is asked to work out the details. Compare *Analytical Conics*, pp. 30–31.

6. Find the equation of the sphere for which the circle

$$x^2 + y^2 + z^2 + 2px + 2qy + 2rz + d = \lambda x + \mu y + \nu z + \rho = 0$$

is a great circle.

7. The tetrahedron $ABCD$ is called an **orthocentric tetrahedron** if the perpendicular from the vertices on the opposite faces are concurrent. Prove that the necessary and sufficient condition is that the three sums of the squares of opposite edges are equal.

If each pair of vertices of a tetrahedron are conjugate points with respect to a sphere, show that the **orthocentre** of the tetrahedron coincides with the centre of the circle.

Answers

1. $x^2 + y^2 + z^2 + 4x - 6y + 2z + 5 = 0$. **2.** $(-5/6, 2/3, -1/6)$, 1. **3.** $9(x^2 + y^2 + z^2) - 17x - 23y - z - 24 = 0$. **7.** 3 or -2. **8.** $x - 5y + z + 12 = 0$, $3x + 11y - 3z - 16 = 0$, $3x - 2y + 10 = 0$. Pencil of planes through $(x + 2)/2 = (y - 2)/3 = z/13$. **9.** $(0, -3, 4)$, $(-2, -1, 0)$, $(-3, 0, -2)$. Collinear on the straight line $x + 2 = -(y + 1) = z/2$. **11.** $x^2 + y^2 + z^2 - 2(x + y + z) - 11 = 0$ and $13(x^2 + y^2 + z^2) - 82(x + y + z) - 87 = 0$. **12.** $\cos^{-1}\{7\sqrt{33}/138\}$. **14.** $(2, 0, -1)$ and $(1, -1, 0)$.

Miscellaneous examples. 1. $2(p_2^2 + q_2^2 + r_2^2) - 2(p_1p_2 + q_1q_2 + r_1r_2) + d_1 - d_2 = 0$.

2.
$$R\begin{pmatrix} \lambda_1 & \mu_1 & \nu_1 & \rho_1 \\ \lambda_2 & \mu_2 & \nu_2 & \rho_2 \\ 2(p_1 - p_2) & 2(q_1 - q_2) & 2(r_1 - r_2) & d_1 - d_2 \end{pmatrix} \leqslant 2.$$

(For explanation of symbolism, see Appendix, p. 116.)

4.
$$(\alpha l + \beta m + \gamma n + pl + qm + rn)^2$$
$$= (l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 + 2p\alpha + 2q\beta + 2r\gamma + d),$$
$$(lx + my + nz - \alpha l - \beta m - \gamma n)^2$$
$$+ (l^2 + m^2 + n^2)(2\alpha x + 2\beta y + 2\gamma z - \alpha^2 - \beta^2 - \gamma^2) = 0.$$

5.
$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ -d_1 & p_1 & q_1 & r_1 & -1 \\ -d_2 & p_2 & q_2 & r_2 & -1 \\ -d_3 & p_3 & q_3 & r_3 & -1 \\ -d_4 & p_4 & q_4 & r_4 & -1 \end{vmatrix} = 0.$$

6.
$$(\lambda^2 + \mu^2 + \nu^2)(x^2 + y^2 + z^2 + 2px + 2qy + 2rz + d)$$
$$+ 2(\rho - \lambda p - \mu q - \nu r)(\lambda x + \mu y + \nu z + \rho) = 0.$$

CHAPTER IV

QUADRIC CONE—CYLINDER

25. Quadric cone

A straight line, called a **generator**, which always passes through a fixed point, called the **vertex**, and which intersects a fixed conic, traces out a surface, called a **quadric cone**. (For brevity we omit the adjective "quadric.") If the conic degenerates to a line-pair, the cone degenerates to a pair of planes or possibly to two coincident planes.

We now obtain the equation of the cone which passes through the conic

$$\psi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0;\ z = 0, \quad (25.1)$$

and has its vertex at A.

Let the direction-ratios of a generator be $\{l, m, n\}$. Any point on the generator has the parametric coordinates $(x_A + lt,\ y_A + mt,\ z_A + nt)$. The point of intersection of this generator and the plane $z = 0$ corresponds to $t = -z_A/n$. This point lies on the conic $\psi(x, y) = 0$ and so $\psi(x_A - lz_A/n,\ y_A - mz_A/n) = 0$. Eliminating l, m, n from this equation and the equations $x = x_A + lt$, $y = y_A + mt$, $z = z_A + nt$ of the generator, we obtain the equation of the cone in the form

$$\psi((z_Ax - x_Az)/(z_A - z),\quad (z_Ay - y_Az)/(z_A - z)) = 0. \quad (25.2)$$

We note that this equation has quadratic terms and linear terms in x, y and z and a constant term.

Example 1. Find the equation of the cone with vertex at $(1, 2, -1)$, which passes through the circle $x^2 + y^2 + z^2 + 2z = x + y + z - 1 = 0$.

26. Cone with vertex at the origin

If the point $A \equiv (x_A, y_A, z_A)$ is on a cone whose vertex is at the origin, then every point on the generator OA is on the cone. That is, the point (kx_A, ky_A, kz_A) is on the cone. Accordingly the equation of the cone must be homogeneous in x, y and z. The terms of highest degree in a cone are quadratic and so the general equation of a cone with its vertex at the origin is

$$\psi(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \qquad (26.1)$$

This equation has five effective constants, namely the ratios of the six quantities a, b, c, f, g, h. Generally, one unique cone can be drawn to pass through five given generators.

If the x-axis is a generator, then (26.1) is identically satisfied by $y = z = 0$. Hence $a = 0$. Consequently a cone with the three coordinate axes as generators has the equation

$$fyz + gzx + hxy = 0.$$

Example 2. Obtain the equation of the cone with vertex at the origin which passes through the circle $x^2 + y^2 + z^2 + 2px + 2qy + 2rz + d = \lambda x + \mu y + \nu z + \rho = 0$.

27. Circular cone

Let P be any point on the right circular cone, semi-vertical angle α, vertex at the origin and whose axis has direction-ratios $\{l, m, n\}$. The direction-ratios of OP are $\{x_P, y_P, z_P\}$ and so

$$\cos\alpha = (lx_P + my_P + nz_P)/\sqrt{(l^2 + m^2 + n^2)(x_P{}^2 + y_P{}^2 + z_P{}^2)}.$$

Thus the equation of the circular cone is

$$(x^2 + y^2 + z^2)(l^2 + m^2 + n^2)\cos^2\alpha - (lx + my + nz)^2 = 0. \quad (27.1)$$

Conversely, the equation

$$x^2 + y^2 + z^2 - (\lambda x + \mu y + \nu z)^2 = 0$$

can be written

$$(x^2 + y^2 + z^2)(\lambda^2 + \mu^2 + \nu^2)/(\lambda^2 + \mu^2 + \nu^2) - (\lambda x + \mu y + \nu z)^2 = 0$$

and so represents a circular cone whose axis has the direction $\{\lambda, \mu, \nu\}$ and whose semi-vertical angle is $\cos^{-1}(\lambda^2 + \mu^2 + \nu^2)^{-\frac{1}{2}}$. This angle is complex if $\lambda^2 + \mu^2 + \nu^2 < 1$ and in this case we say that the cone is **virtual.**

28. Conditions that $\psi, (x, y, z)$ be a circular cone

If equation (26.1) represents a circular cone it can be identified with (27.1). Comparing coefficients we have

$$\begin{aligned} ka &= (l^2 + m^2 + n^2)\cos^2\alpha - l^2; & kf &= -mn; \\ kb &= (l^2 + m^2 + n^2)\cos^2\alpha - m^2; & kg &= -nl; \\ kc &= (l^2 + m^2 + n^2)\cos^2\alpha - n^2; & kh &= -lm; \end{aligned}$$

where k is some factor of proportionality. If $fgh \neq 0$, we easily obtain

$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}. \tag{28.1}$$

If $f = 0$, then either $m = 0$ or $n = 0$ and so either $h = 0$ or $g = 0$. Let us suppose $m = 0$. Then $f = h = 0$. We now find that the condition for a circular cone is

$$g^2 = (a - b)(c - b). \tag{28.2}$$

Finally if $f = g = h = 0$, then two of l, m and n must be zero, say $l = m = 0$. In this case the required condition reduces to $a = b$.

The reader can verify that all these cases can be combined into the statement that the rank of the matrix†

$$\begin{pmatrix} a - \lambda & h & g \\ h & b - \lambda & f \\ g & f & c - \lambda \end{pmatrix}$$

is one.

Example 3. If $fgh \neq 0$, show that the circular cone (26.1) has its axis in the direction $\{1/f,\ 1/g,\ 1/h\}$ and semi-vertical angle given by $3\cos^2\alpha = 1 - fgh(a + b + c)/(g^2h^2 + h^2f^2 + f^2g^2)$.

Example 4. If $f = h = 0$, show that the circular cone (26.1) has its axis in the direction $\{\sqrt{b - a},\ 0,\ \sqrt{b - c}\}$ and the semi-vertical angle is given by $3\cos^2\alpha = -3b/(a - 2b + c)$.

Example 5. If $f = g = h = 0$ and $a = b$, show that the axis is in the direction $\{0, 0, 1\}$ and the semi-vertical angle is given by $\tan^2\alpha = -c/a$.

29. Angle between two generators

We now determine the angle θ between the two generators of the cone (26.1) which lie in the plane

$$\lambda x + \mu y + \nu z = 0. \tag{29.1}$$

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† See Appendix, p. 116.

Let the equation of one of these generators be $x/l = y/m = z/n$. Any point on it has the parametric coordinates (lt, mt, nt). Since this point lies on (26.1) and (29.1) we have

$$\psi(l, m, n) \equiv al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0,$$
$$\lambda l + \mu m + \nu n = 0.$$

There is no loss in generality if we assume $n \neq 0$. The elimination of n yields

$$l^2(a\nu^2 - 2g\lambda\nu + c\lambda^2) + 2lm(h\nu^2 + c\lambda\mu - g\mu\nu - f\lambda\nu) \\ + m^2(b\nu^2 - 2f\mu\nu + c\mu^2) = 0.$$

Let the direction-ratios of the two generators be $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$. From the theory of quadratic equations, it follows that

$$\frac{l_1l_2}{b\nu^2 - 2f\mu\nu + c\mu^2} = \frac{m_1m_2}{c\lambda^2 - 2g\lambda\nu + a\nu^2} \\ = \frac{l_1m_2 + l_2m_1}{-2(h\nu^2 + c\lambda\mu - g\mu\nu - f\lambda\nu)} = \kappa \text{ (say)}.$$

From symmetry we also have

$$\frac{n_1n_2}{a\mu^2 - 2h\lambda\mu + b\lambda^2} = \frac{m_1n_2 + m_2n_1}{-2(f\lambda^2 + a\mu\nu - h\lambda\nu - g\lambda\mu)} \\ = \frac{n_1l_2 + n_2l_1}{-2(g\mu^2 + b\lambda\nu - f\lambda\mu - h\mu\nu)} = \kappa.$$

In virtue of (6.2) and (6.5) we have

$$\tan\theta = \pm\{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 \\ + (n_1l_2 - l_2n_1)^2\}^{\frac{1}{2}}/(l_1l_2 + m_1m_2 + n_1n_2).$$

But

$$l_1l_2 + m_1m_2 + n_1n_2 = \kappa\{(a + b + c)(\lambda^2 + \mu^2 + \nu^2) - \psi(\lambda, \mu, \nu)\}$$

and a calculation yields

$$(l_1m_2 - l_2m_1)^2 = (l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2 = -4\kappa^2\nu^2\phi(\lambda, \mu, \nu)$$

where

$$\left.\begin{aligned}\phi(\lambda, \mu, \nu) &\equiv (bc - f^2)\lambda^2 + (ca - g^2)\mu^2 + (ab - h^2)\nu^2 \\ &\quad + 2(hg - af)\mu\nu + 2(hf - bg)\nu\lambda + 2(fg - hc)\lambda\mu \\ &\equiv \mathscr{A}\lambda^2 + \mathscr{B}\mu^2 + \mathscr{C}\nu^2 + 2\mathscr{F}\mu\nu + 2\mathscr{G}\nu\lambda + 2\mathscr{H}\lambda\mu\end{aligned}\right\} \quad (29.2)$$

if we use the notation of *Analytical Conics*,† p. 76.

Hence

$$\tan\theta = \pm\frac{2\{-(\lambda^2 + \mu^2 + \nu^2)\phi(\lambda, \mu, \nu)\}^{\frac{1}{2}}}{(a + b + c)(\lambda^2 + \mu^2 + \nu^2) - \psi(\lambda, \mu, \nu)}. \quad (29.3)$$

Thus the generators are real, coincident or conjugate complex according as $\phi(\lambda, \mu, \nu)$ is less than, equal to or greater than zero respectively. For $\phi(\lambda, \mu, \nu) = 0$, the plane $\lambda x + \mu y + \nu z = 0$ is a tangent plane.

Later we shall require the particular result that $\lambda x + \mu y + \nu z = 0$ is a tangent plane to the cone $ax^2 + by^2 + cz^2 = 0$, if

$$\frac{\lambda^2}{a} + \frac{\mu^2}{b} + \frac{\nu^2}{c} = 0. \quad (29.4)$$

Further, we deduce that the pair of generators are orthogonal if

$$(a + b + c)(\lambda^2 + \mu^2 + \nu^2) - \psi(\lambda, \mu, \nu) = 0. \quad (29.5)$$

Example 6. Calculate the angle between the two generators of the cone $yz + zx + xy = 0$ which lie in the plane $6x + 3y - 2z = 0$.

Example 7. Show that the plane $x - y + 2z = 0$ cuts the cones $x^2 - 3y^2 + 2z^2 = 0$ and $5yz + 3zx + 2xy = 0$ in the same pair of generators which intersect at an angle $\tan^{-1}\{4\sqrt{6}/3\}$. Further show that the other common generators of the two cones are complex.

30. Reciprocal cone

The plane $\lambda x + \mu y + \nu z = 0$ is a tangent plane to the cone $\psi(x, y, z) = 0$ if $\phi(\lambda, \mu, \nu) = 0$. The line $x/\lambda = y/\mu = z/\nu$ is normal to this plane and so it generates the cone $\phi(x, y, z) = 0$ called the **reciprocal** cone.

Example 8. Show that the reciprocal cone of the reciprocal cone is the original cone.

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† Published by Pergamon Press, London, 1957.

31. Rectangular and orthogonal cones

The plane $\lambda x + \mu y + \nu z = 0$ cuts the cone $\psi(x, y, z) = 0$ in orthogonal generators if equation (29.4) is satisfied. The normal to this plane is also a generator if $\psi(\lambda, \mu, \nu) = 0$. Thus if three mutually orthogonal generators exist, then

$$a + b + c = 0. \tag{31.1}$$

Conversely, if $a + b + c = 0$, we shall now show that the cone possesses an infinite number of mutually orthogonal triples of generators. Let any generator be $x/\lambda = y/\mu = z/\nu$. Then $\psi(\lambda, \mu, \nu) = 0$. Thus equation (29.5) is satisfied. Hence the plane $\lambda x + \mu y + \nu z = 0$ cuts the cone in orthogonal generators. Consequently any generator gives rise to a triple of mutually orthogonal generators. The problem of finding three mutually orthogonal generators of a cone is thus **poristic.** If (31.1) is satisfied we say that the cone is **rectangular.**

The reciprocal of a rectangular cone possesses an infinite number of triples of mutually orthogonal tangent planes. Such a cone is called an **orthogonal** cone. The reciprocal of an orthogonal cone is a rectangular cone and so in virtue of (29.2) the necessary and sufficient condition that the cone $\psi(x, y, z) = 0$ be orthogonal is

$$\mathscr{A} + \mathscr{B} + \mathscr{C} \equiv bc + ca + ab - f^2 - g^2 - h^2 = 0. \tag{31.2}$$

It will be clear from §77 on relative invariants that equations (31.1) and (31.2) are still valid if there are additional linear terms in the equation of the cone. (This will be useful in solving Examples 3, pages 41 and 54.)

Example 9. Prove that the semi-vertical angle of a circular cone is $\tan^{-1} 2$ if the cone is rectangular but $\cot^{-1} 2$ if the cone is orthogonal.

32. Quadric cylinder

A straight line of fixed direction, called a **generator,** which always intersects a conic traces out a surface, called a **quadric cylinder.** (For brevity, we omit the adjective "quadric.") If the conic degenerates to a line-pair, the cylinder degenerates to a pair of planes or possibly to two coincident planes.

We now find the equation of the cylinder whose generators are parallel to the direction $\{l, m, n\}$ and which has the section (25.1).

Let A be any point on this section. The generator of the cylinder through A is $(x - x_A)/l = (y - y_A)/m = z/n$. Further $\psi(x_A, y_A) = 0$

and the elimination of x_A and y_A yields the equation of the cylinder in the form $\psi(x - lz/n, y - mz/n) = 0$. If we choose the generators parallel to the z-axis, then $l = m = 0$ and the cylinder is represented by the equation $\psi(x, y) = 0$. We then say that the cylinder is elliptic, hyperbolic or parabolic according as $\psi(x, y) = z = 0$ is an ellipse, hyperbola or parabola. Their standard forms can then be taken respectively as $x^2/a^2 + y^2/b^2 = 1$, $x^2/a^2 - y^2/b^2 = 1$ and $y^2 = 4ax$.

Example 10. Obtain the equation of the cylinder whose generators are parallel to the direction $\{1, 2, -1\}$ and which has the circular section $x^2 + y^2 + z^2 + 2z = x + y + z - 1 = 0$.

Miscellaneous examples

1. Show that the general equation of a cone which touches the three coordinate planes is $a^2x^2 + b^2y^2 + c^2z^2 \pm 2bcyz \pm 2cazx \pm 2abxy = 0$. (The choice of signs is restricted to the cases where the expression is *not* a perfect square.)

2. Obtain the conditions that the cones $fyz + gzx + hxy = 0$ and $ax^2 + by^2 + cz^2 = 0$ have a common pair of generators lying in the plane $\lambda x + \mu y + \nu z = 0$.

3. Prove that $ax^2 + by^2 + cz^2 + 2px + 2qy + 2rz + d = 0$ represents a cone if $p^2/a + q^2/b + r^2/c = d$. Further, find the coordinates of the vertex.

4. Show that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$ represents a circular cone of semi-vertical angle $\sin^{-1} 1/\sqrt{3}$.

5. Prove that the locus of a point whose distance from a fixed plane equals its distance from a fixed line is a cone.

6. Show that the locus of the mid-points of the intercepts between two fixed planes of a straight line through a fixed point is a hyperbolic cylinder.

7. Find the locus of points, with real coordinates, which are the vertices of right circular cones having the elliptical section $x^2/a^2 + y^2/b^2 - 1 = z = 0$ $(a > b)$.

Answers

1. $3x^2 + y^2 + 5z^2 + 4yz + 6zx + 2xy - 4x - 2y - 4z + 2 = 0$.
2. $\rho^2(x^2 + y^2 + z^2) - 2\rho(\lambda x + \mu y + \nu z)(px + qy + rz) + d(\lambda x + \mu y + \nu z)^2 = 0$. **6.** $\pi/2$. **10.** $3x^2 + y^2 + 7z^2 + 4yz + 6zx - 2x - 2z + 1 = 0$.

Miscellaneous examples. 2. $(b\nu^2 + c\mu^2)/f\mu\nu = (c\lambda^2 + a\nu^2)/g\nu\lambda = (a\mu^2 + b\lambda^2)/h\lambda\mu$. **3.** $(-p/a, -q/b, -r/c)$. **7.** $x^2/(a^2 - b^2) - z^2/b^2 - 1 = y = 0$.

CHAPTER V

CENTRAL QUADRIC

33. Central quadric

In plane geometry we discussed the central conics with the equation $ax^2 + by^2 + c = 0$. This suggests that we discuss the surfaces, called **central quadrics,** represented by the equation

$$ax^2 + by^2 + cz^2 + d = 0. \tag{33.1}$$

If the product $abcd \neq 0$, we say that the central quadric is non-degenerate. The degenerate cases can be listed as follows:

(1) Cone, vertex at the origin: $d = 0$.

(2) Elliptic or hyperbolic cylinder: $a = 0$ or $b = 0$ or $c = 0$.

(3) Pair of intersecting planes: $d = a = 0$ or $d = b = 0$ or $d = c = 0$.

(4) Pair of parallel planes: $b = c = 0$ or $c = a = 0$ or $a = b = 0$.

(5) Pair of coincident planes: $d = b = c = 0$ or $d = c = a = 0$ or $d = a = b = 0$.

Equation (33.1) contains only even powers of x, y and z and so a central quadric is symmetrical with respect to the coordinate planes. Further the origin, called the **centre,** bisects all chords, called **diameters,** which pass through it.

The straight line through A in the direction $\{l, m, n\}$ intersects the central quadric (33.1) at the points t corresponding to the solution of the quadratic equation

$$a(x_A + lt)^2 + b(y_A + mt)^2 + c(z_A + nt)^2 + d = 0. \tag{33.2}$$

Consequently, a straight line intersects a central quadric in two points which may be real and distinct, real and coincident or conjugate complex. However, the quadratic equation may become an identity in t and then the straight line lies completely in the surface and is called a **generator.**

Any plane section of a central quadric is a conic, since any straight line, not a generator, in the plane of the section cuts the surface and therefore also the section in *two* points.

Excluding the degenerate cases, we may write (33.1) in the form

$$-(a/d)x^2 - (b/d)y^2 - (c/d)z^2 = 1. \tag{33.3}$$

We now proceed to discuss the different cases which arise, according to the signs of the coefficients of x^2, y^2 and z^2.

34. Ellipsoid

If the coefficient of x^2, y^2 and z^2 in (33.3) are all negative, then the surface represented by (33.3) cannot have any real points. We say that the surface is a **virtual ellipsoid.**

Next, we consider the case when all the coefficients are positive. We may rewrite (33.3) in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{34.1}$$

It is clear that this surface, called an **ellipsoid** (Fig. 14), lies completely within the rectangular parallelepiped formed by the planes $x = \pm a$, $y = \pm b$ and $z = \pm c$. Further, all real sections parallel to the coordinate planes are ellipses. In fact all real sections are ellipses since the surface is finite in extent.

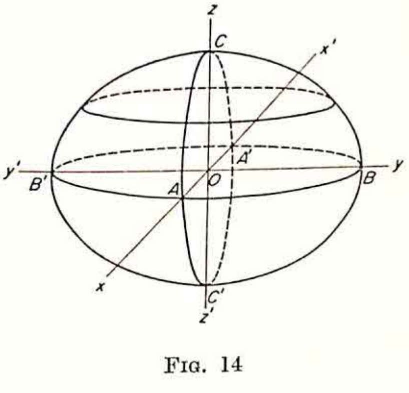

FIG. 14

If two of a, b, c are equal, the surface is of revolution whilst if $a = b = c$ the surface is a sphere.

35. Hyperboloid of one sheet

We now consider the case when equation (33.3) has one negative and two positive coefficients. We may write (33.3)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{35.1}$$

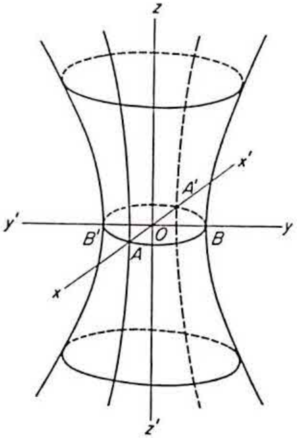

FIG. 15

All sections parallel to $z = 0$ are real ellipses with a minimum section corresponding to $z = 0$, whilst sections parallel to $x = 0$ and $y = 0$ are hyperbolae.

The section of (35.1) by $x = a$ is the pair of generators $x = a$, $y/b + z/c = 0$ and $x = a$, $y/b - z/c = 0$. The section by $x = -a$ is also a pair of generators. The value of $y^2/b^2 - z^2/c^2 = 1 - x^2/a^2$ is positive for $|x| < a$ but negative if $|x| > a$. Hence, the directions of the conjugate and transverse axes of the hyperbolic sections parallel to the plane $x = 0$ interchange on passing through $x = \pm a$. In the same way the directions of the conjugate and transverse axes of the hyperbolic sections parallel to the plane $y = 0$ interchange on passing through $y = \pm b$.

This surface is called a **hyperboloid of one sheet** (Fig. 15). It is a surface of revolution if $a = b$.

36. Hyperboloid of two sheets

Finally, we examine the case when equation (33.3) has one positive and two negative coefficients. We may then write this equation in the form

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{36.1}$$

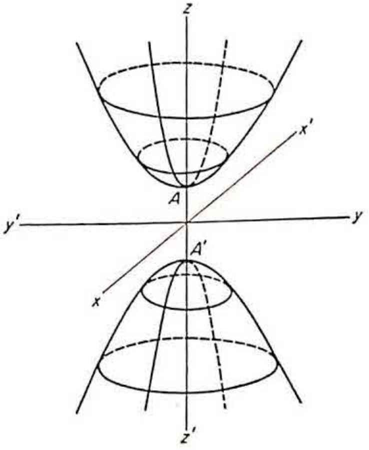

FIG. 16

This surface has no real points in the domain between the planes $z=c$ and $z=-c$. All sections by planes parallel to $z=0$ but outside the domain $-c\leq z\leq c$ are real ellipses. Further, the sections parallel to $x=0$ and $y=0$ are hyperbolae. Since $-y^2/b^2+z^2/c^2=1+x^2/a^2$ does not change sign with x, there is no interchange of the transverse and conjugate axes of the hyperbolic sections as occurs in the hyperboloid of one sheet.

The surface (36.1) consists of two disconnected parts and is called the **hyperboloid of two sheets** (Fig. 16). It is a surface of revolution if $a=b$.

37. Polar properties

Consider the central quadrics represented by equation (33.1). For brevity, let us write

$$F_A\equiv ax_A{}^2+by_A{}^2+cz_A{}^2+d;\quad F_B\equiv ax_B{}^2+by_B{}^2+cz_B{}^2+d;$$

$$T_{AB}=T_{BA}\equiv ax_Ax_B+by_Ay_B+cz_Az_B+d;$$

$$T_A\equiv ax_Ax+by_Ay+cz_Az+d.$$

The coordinates of the point P which divides AB in the ratio $\sigma:\tau$ are given by (3.1). A simple calculation shows that P lies on the quadric (33.1) if

$$\tau^2 F_A + 2\tau\sigma T_{AB} + \sigma^2 F_B = 0. \tag{37.1}$$

This equation is called Joachimsthal's quadratic equation (compare §19), and its roots correspond to the two points of intersection P and Q of AB and the central quadric.

Two points A and B are said to be conjugate with respect to the central quadric if the points of intersection P and Q divide A and B harmonically. As in §19, the necessary and sufficient condition is $T_{AB} = 0$.

The **polar** surface of a point is defined to be the locus of all points conjugate to it. Consequently the polar surface of A is the plane $T_A = 0$.

The **pole** of a plane is defined to be that point whose polar is the given plane. Let us now find the coordinates of the pole A of the plane $\lambda x + \mu y + \nu z + \rho = 0$. The polar of A is the plane $T_A = 0$ and the comparison of coefficients yields

$$ax_A/\lambda = by_A/\mu = cz_A/\nu = d/\rho.$$

Accordingly the pole A is at the point $(\lambda d/a\rho, \mu d/b\rho, \nu d/c\rho)$.

It follows that the plane $u_1 \equiv \lambda_1 x + \mu_1 y + \nu_1 z + \rho_1 = 0$ passes through the pole of the plane $u_2 \equiv \lambda_2 x + \mu_2 y + \nu_2 z + \rho_2 = 0$ if

$$\lambda_1\lambda_2/a + \mu_1\mu_2/b + \nu_1\nu_2/c + \rho_1\rho_2/d = 0. \tag{37.2}$$

From the symmetry in the indices 1 and 2, we see that the plane $u_2 = 0$ also passes through the pole of the plane $u_1 = 0$. Two planes such that each passes through the pole of the other are said to be **conjugate.**

Now let the point A lie on the quadric. That is, $F_A = 0$. Equation (37.1) has coincident roots if $T_{AB} = 0$. Consequently the locus of the point B, such that AB is a tangent line, is the plane $T_A = 0$ called the **tangent** plane. It is identical in form with the polar of A. Thus the pole of a tangent plane is at its point of contact.

A tangent plane is clearly conjugate to itself. Accordingly from (37.2) the plane $\lambda x + \mu y + \nu z + \rho = 0$ is a tangent plane if

$$\lambda^2/a + \mu^2/b + \nu^2/c + \rho^2/d = 0. \tag{37.3}$$

Return again to equation (37.1) and let AB be a tangent line with neither A nor B on the quadric. Then the roots of (37.1) are coincident and so $F_A F_B - T_{AB}^2 = 0$. Hence the locus of the point B if A is held fixed is the surface given by

$$F_A F - T_A^2 = 0. \tag{37.4}$$

This surface is a cone with vertex at A. Since (37.4) is quadratic in x, y and z the surface is a quadric cone, called the **enveloping cone** or **tangent cone** from A.

Example 1. Show that the plane

$$\lambda x + \mu y + \nu z = \pm \sqrt{-d(\lambda^2/a + \mu^2/b + \nu^2/c)}$$

is always a tangent plane of the quadrics $ax^2 + by^2 + cz^2 + d = 0$.

Example 2. Prove that the locus of the point of intersection of three mutually orthogonal tangent planes of the quadrics $ax^2 + by^2 + cz^2 + d = 0$ is the so-called **orthoptic sphere** $x^2 + y^2 + z^2 + d(1/a + 1/b + 1/c) = 0$.

Example 3. Show that the locus of points from which three mutually orthogonal tangent lines can be drawn to the quadrics $ax^2 + by^2 + cz^2 + d = 0$ is the quadric $a(b + c)x^2 + b(c + a)y^2 + c(a + b)z^2 + d(a + b + c) = 0$.

Example 4. Show that the locus of the pole with respect to $ax^2 + by^2 + cz^2 + d = 0$ of tangent planes to $\alpha x^2 + \beta y^2 + \gamma z^2 + \delta = 0$ is the quadric $a^2x^2/\alpha + b^2y^2/\beta + c^2z^2/\gamma + 1/\delta = 0$.

38. Generators

Equation (37.1) may be an identity. If so, the line is a generator. The necessary and sufficient conditions are

$$F_A = 0; \quad F_B = 0; \quad T_{AB} = 0. \tag{38.1}$$

The parametric coordinates of any point on the line AB are given by (9.4) and so the tangent plane at this point is

$$\xi(x, y, z) \equiv a\{(1 - t)x_A + tx_B\}x + b\{(1 - t)y_A + ty_B\} \\ + c\{(1 - t)z_A + tz_B\}z + d = 0.$$

We readily verify that

$$\xi\big((1 - s)x_A + sx_B, \quad (1 - s)y_A + sy_B, \quad (1 - s)z_A + sz_B\big) \equiv 0$$

in virtue of (38.1). Hence a generator lies completely in the tangent plane at any point on it. The section of the quadric by the tangent plane must now be a line-pair. Therefore, in general two generators pass through a point and they determine the tangent plane at that point.

From equations (38.1) we have

$$(ax_A{}^2 + by_A{}^2)(ax_B{}^2 + by_B{}^2) - (ax_Ax + by_Ay)^2 \\ = (cz_A{}^2 + d)(cz_B{}^2 + d) - (cz_Az_B + d)^2.$$

That is,

$$ab(x_Ay_B - x_By_A)^2 = cd(z_A - z_B)^2.$$

Thus the generators are real if the product *abcd* is positive. If a, b, c, d all have the same sign the quadric (33.1) is virtual and cannot have real generators. Hence two of a, b, c, d are positive and two are negative. This corresponds to a hyperboloid of one sheet and so this surface is the only non-degenerate central quadric with real generators. All the degenerate central quadrics have generators.

39. Generators of hyperboloid of one sheet

The equation (35.1) of the hyperboloid may be written

$$\left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right).$$

Hence the generators are the lines of intersection of the planes

$$\frac{x}{a} + \frac{z}{c} = \lambda\left(1 + \frac{y}{b}\right); \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{\lambda}\left(1 - \frac{y}{b}\right) \tag{39.1}$$

or the planes

$$\frac{x}{a} - \frac{z}{c} = \mu\left(1 + \frac{y}{b}\right); \quad \frac{x}{a} + \frac{z}{c} = \frac{1}{\mu}\left(1 - \frac{y}{b}\right). \tag{39.2}$$

The generators (39.1) and (39.2) may be called respectively the λ-system and the μ-system.

Two *distinct* generators of the λ-system are given by

$$\left.\begin{aligned} x/a + z/c = \lambda_1(1 + y/b);\quad x/a - z/c = (1 - y/b)/\lambda_1 \\ x/a + z/c = \lambda_2(1 + y/b);\quad x/a - z/c = (1 - y/b)/\lambda_2 \end{aligned}\right. \quad (\lambda_1 \neq \lambda_2).$$

These four equations are inconsistent. It follows that no two generators of the λ-system intersect. Similarly all the generators of the μ-system are skew to one another.

We readily verify that the point

$$\left(\frac{a(\lambda + \mu)}{1 + \lambda\mu}, \ \frac{b(1 - \lambda\mu)}{1 + \lambda\mu}, \ \frac{c(\lambda - \mu)}{1 + \lambda\mu}\right) \tag{39.3}$$

lies on both generators (39.1) and (39.2). Hence each generator of the λ-system intersects every generator of the μ-system.

Example 5. Show that the locus of the points of intersection of orthogonal generators is the curve of intersection of the hyperboloid and its orthoptic sphere. (See Example 2, §37.)

Example 6. Calculate the angle between the two generators of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ which pass through the point $(a \cos \alpha, b \sin \alpha, 0)$.

40. Normals

The normal at A has the equations

$$\frac{x - x_A}{ax_A} = \frac{y - y_A}{by_A} = \frac{z - z_A}{cz_A}. \tag{40.1}$$

We ask: "How many normals pass through the fixed point K?" The answer depends on the number of solutions for A of the equations

$$\frac{x_K - x_A}{ax_A} = \frac{y_K - y_A}{by_A} = \frac{z_K - z_A}{cz_A} = t \text{ (say)} \tag{40.2}$$

and $F_A = 0$. Introducing the parameter t in (40.2) we obtain

$$x_A = x_K/(1 + at); \quad y_A = y_K/(1 + bt); \quad z_A = z_K/(1 + ct) \tag{40.3}$$

and so by substitution in $F_A = 0$ we have

$$\frac{ax_K^2}{(1 + at)^2} + \frac{by_K^2}{(1 + bt)^2} + \frac{cz_K^2}{(1 + ct)^2} + d = 0. \tag{40.4}$$

This equation is of the sixth degree in t. Hence six normals can be drawn from a point of general position to a central quadric.

If the quadric is of revolution $(a = b)$ or is a cone $(d = 0)$, equation (40.4) reduces to a quartic and so only four normals can be drawn from a point to a central quadric of revolution or to a cone.

We now show that the six normals through K lie on a quadric cone. This cone is obtained by eliminating x_A, y_A, z_A and t from the equations

$$\frac{x - x_K}{ax_A} = \frac{y - y_K}{by_A} = \frac{z - z_K}{cz_A}$$

of the normal KA and equations (40.3). The result is

$$\begin{aligned} a(b - c)x_K(y - y_K)(z - z_K) + b(c - a)y_K(z - z_K)(x - x_K) \\ + c(a - b)z_K(x - x_K)(y - y_K) = 0. \end{aligned} \tag{40.5}$$

Example 7. Show that the six normals drawn to the quadric $ax^2 + by^2 + cz^2 + d = 0$ from any point on one of the straight lines $a(b - c)x = \pm b(c - a)y = \pm c(a - b)z$ lie on a right circular cone.

Example 8. Prove that the locus of the mid-points of parallel chords of a quadric, such that the normals at the ends of a chord intersect, is a straight line.

41. Centre of a section

The centre of a plane section of a central quadric is that point in the plane which bisects all chords of the plane which pass through it. Let us find the coordinates of the centre A of the section of the central quadric (33.1) by the plane $\lambda x + \mu y + \nu z + \rho = 0$.

The straight line through A in the direction $\{l, m, n\}$ has the equations (9.1). Its points of intersection with the central quadric (33.1) are determined by the roots of the quadratic equation (33.2) in t. Hence A is the centre of the chord through A in the direction $\{l, m, n\}$ if (33.2) has equal and opposite roots in t. That is,

$$alx_A + bmy_A + cnz_A = 0. \tag{41.1}$$

Accordingly, the equation of the plane whose centre is at A is obtained by eliminating l, m, n from (41.1) and (9.1). The result is

$$ax_A(x - x_A) + by_A(y - y_A) + cz_A(z - z_A) = 0.$$

This equation now represents the same plane as $\lambda x + \mu y + \nu z + \rho = 0$, and so

$$\frac{ax_A}{\lambda} = \frac{by_A}{\mu} = \frac{cz_A}{\nu} = -\frac{ax_A{}^2 + by_A{}^2 + cz_A{}^2}{\rho}.$$

To solve these equations, put each fraction equal to t. Then $x_A = \lambda t/a$, etc. Further $(\lambda^2/a + \mu^2/b + \nu^2/c)t + \rho = 0$. Consequently

$$A \equiv (-\lambda\rho/a\kappa, \quad -\mu\rho/b\kappa, \quad -\nu\rho/c\kappa) \tag{41.2}$$

where

$$\kappa = \lambda^2/a + \mu^2/b + \nu^2/c. \tag{41.3}$$

This section is a parabola if $\kappa = 0$ but $\rho \neq 0$. In this case at least one of a, b or c must be negative. Thus the hyperboloids are the only non-degenerate central quadrics with parabolic sections.

Example 9. Show that the locus of the centres of parallel plane sections of a quadric is a diameter.

Example 10. Prove that the locus of the centres of sections of $ax^2 + by^2 + cz^2 + d = 0$ by tangent planes of the quadric $\alpha x^2 + \beta y^2 + \gamma z^2 + \delta = 0$ is $\delta(a^2x^2/\alpha + b^2y^2/\beta + c^2z^2/\gamma) + (ax^2 + by^2 + cz^2)^2 = 0$.

42. Axes of central plane section

Equal diameters of a central conic are equally inclined to the axes. Thus we can determine the axes as the limiting positions of equal diameters when they tend to coincide with one another.

Let us consider the diameter of length $2R$ in the direction $[l, m, n]$ which lies in the given central plane $\lambda x + \mu y + \nu z = 0$ of the central quadric (33.1). The point (lR, mR, nR) lies on the quadric and so $(al^2 + bm^2 + cn^2)R^2 + d = 0$. This equation may be written

$$(aR^2 + d)l^2 + (bR^2 + d)m^2 + (cR^2 + d)n^2 = 0 \qquad (42.1)$$

since l, m, n are direction-cosines. Further (lR, mR, nR) lies in the given plane. Therefore,

$$\lambda l + \mu m + \nu n = 0. \qquad (42.2)$$

The axes of the section correspond to the case when these two equations have coincident roots in $l{:}m{:}n$. We may eliminate one of l, m, n and write down the condition that the resulting quadratic has equal roots. It is more instructive to regard for the moment l, m, n as coordinates. With this interpretation the plane (42.2) is a tangent plane to the cone (42.1). In virtue of (29.4) the required condition is

$$\frac{\lambda^2}{aR^2 + d} + \frac{\mu^2}{bR^2 + d} + \frac{\nu^2}{cR^2 + d} = 0. \qquad (42.3)$$

This equation is quadratic in R^2 and its roots correspond to the squares of the lengths of the semi-axes. The roots are both positive for a real elliptic section, both negative for a virtual section whilst for a hyperbolic section one root is positive and the other is negative.

The coefficient of R^4 in (42.3) after multiplication by

$$(aR^2 + d)(bR^2 + d)(cR^2 + d)$$

is $abc(\lambda^2/a + \mu^2/b + \nu^2/c)$ which can only be zero if $\lambda x + \mu y + \nu z = 0$ is a tangent plane to the central quadric. Thus the coefficient of R^4 cannot vanish in the case of a central plane section.

The direction of the axes is obtained by observing that the solution in l, m, n of (42.1) and (42.2) subject to (42.3) is

$$l{:}m{:}n = \frac{\lambda}{aR^2 + d} : \frac{\mu}{bR^2 + d} : \frac{\nu}{cR^2 + d}. \qquad (42.4)$$

Example 11. Show that tangent planes to the cone $x^2/(b+c)+y^2/(c+a)+z^2/(a+b)=0$ cut the quadric $ax^2+by^2+cz^2+d=0$ in rectangular hyperbolae.

Example 12. Show that the hyperboloid $x^2+3y^2-4z^2=1$ is cut in circular sections by the planes $2y^2-5z^2=0$.

43. Axes of non-central plane section

The reader is asked at this stage to read §73 on the translation of axes. We now wish to obtain the axes of the section of the central quadric (33.1) by the plane $\lambda x+\mu y+\nu z+\rho=0$, $(\rho\neq 0)$. Let us translate axes to the centre of this section. The coordinates of the centre are given by (41.2) and so the equations specifying the translation are

$$x=X-\lambda\rho/a\kappa;\quad y=Y-\mu\rho/b\kappa;\quad z=Z-\nu\rho/c\kappa.$$

The plane becomes the central plane $\lambda X+\mu Y+\nu Z=0$ whilst the equation of the quadric becomes

$$a(X-\lambda\rho/a\kappa)^2+b(Y-\mu\rho/b\kappa)^2+c(Z-\nu\rho/c\kappa)^2+d=0.$$

In virtue of $\lambda X+\mu Y+\nu Z=0$ and the value of κ as given by (41.3) this equation reduces to

$$aX^2+bY^2+cZ^2+d+\rho^2/\kappa=0. \tag{43.1}$$

Hence the section we are investigating is the same as the section of the central quadric (43.1) by a *central* plane. Accordingly by the previous section, the squares of the lengths of the semi-axes are the roots in R^2 of the quadratic equation

$$\frac{\lambda^2}{aR^2+d+\rho^2/\kappa}+\frac{\mu^2}{bR^2+d+\rho^2/\kappa}+\frac{\nu^2}{cR^2+d+\rho^2/\kappa}=0, \tag{43.2}$$

and the direction-ratios of the axes, which are not affected by the translation, are

$$\{\lambda/(aR^2+d+\rho^2/\kappa),\quad \mu/(bR^2+d+\rho^2/\kappa),\quad \nu/(cR^2+d+\rho^2/\kappa)\}.$$

If $\kappa=0$ the section is a parabola and the calculations of this section are clearly not applicable.

Example 13. Show that the hyperboloid $x^2+3y^2-4z^2=1$ is cut in circular sections by any plane parallel to the planes $2y^2-5z^2=0$.

Example 14. Prove that the area of the section of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ by the plane $x/a+y/b+z/c=1$ is $2\pi(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}/3\sqrt{3}$.

Miscellaneous examples

1. Show that the two tangent planes through the straight line $u_i \equiv \lambda_i x + \mu_i y + \nu_i z + \rho_i = 0$ $(i = 1, 2)$ to the quadric $ax^2 + by^2 + cz^2 + d = 0$ are given by $u_1^2 \Sigma \lambda_2^2/a - 2u_1 u_2 \Sigma \lambda_1 \lambda_2/a + u_2^2 \Sigma \lambda_1^2/a = 0$.

2. Show that the straight line $(x - x_A)/l = (y - y_A)/m = (z - z_A)/n$ touches the quadric $ax^2 + by^2 + cz^2 + d = 0$ if $(ax_A l + by_A m + cz_A n)^2 = (al^2 + bm^2 + cn^2)(ax_A^2 + by_A^2 + cz_A^2 + d)$.

3. Find the locus of the centres of sections of the quadric $ax^2 + by^2 + cz^2 + d = 0$ which are at a constant distance p from the centre.

4. Show that the plane $\lambda x + \mu y + \nu z + \rho = 0$ cuts the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ in a parabola if $\lambda^2 a^2 + \mu^2 b^2 - \nu^2 c^2 = 0$.

5. Prove that the locus of the points of intersection of the normals to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at points of a fixed generator with the plane $z = 0$ is a straight line. Further, show that this straight line envelops a conic as the generator varies.

6. Find the equation of the plane section of the quadric $ax^2 + by^2 + cz^2 + d = 0$ which has one of its axes along the straight line $x/l = y/m = z/n$.

7. Show that the plane determined by two parallel generators of a hyperboloid passes through the centre.

Answers

5. $\cos^{-1} \{(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha - c^2)/(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha + c^2)\}$.

Miscellaneous examples. **3.** $(ax^2 + by^2 + cz^2)^2 = p^2(a^2x^2 + b^2y^2 + c^2z^2)$. **5.** $x^2/(a + c^2/a)^2 + y^2/(b + c^2/b)^2 = 1$. **6.** $(l^2 + m^2 + n^2)(alx + bmy + cnz) - (al^2 + bm^2 + cn^2)(lx + my + nz) = 0$.

CHAPTER VI

PARABOLOID

44. Paraboloid

In this chapter we discuss the paraboloids defined by

$$ax^2 + by^2 + 2rz = 0. \tag{44.1}$$

If the product $abr \neq 0$, we say that the paraboloid is non-degenerate. The degenerate cases are:

(1) Parabolic cylinder: $a = 0$ or $b = 0$.
(2) Pair of intersecting planes: $r = 0$.
(3) Pair of coincident planes: $a = r = 0$ or $b = r = 0$.

Equation (44.1) contains even powers of x and y but not of z. Thus a paraboloid is only symmetric with respect to the planes $x = 0$ and $y = 0$. There is no centre in the usual sense and the origin is now called the **vertex,** whilst the z-axis is called the **axis.**

The straight line through A in the direction $\{l, m, n\}$ intersects the paraboloid (44.1) at the points t corresponding to the solution of the quadratic equation

$$a(x_A + lt)^2 + b(y_A + mt)^2 + 2r(z_A + nt) = 0. \tag{44.2}$$

Consequently a straight line intersects a paraboloid in two points which may be real and distinct, real and coincident or conjugate complex. If (44.2) is an identity then the straight line is a generator. Further, we note that (44.2) becomes linear if $l = m = 0$. That is, straight lines parallel to the z-axis intersect (44.1) in only *one* finite point.

As for the central quadrics, it follows that all plane sections of a paraboloid are conics.

Two cases of (44.1) arise according as a and b have the same or opposite signs.

Case 1. If a and b have the same sign, we may rewrite (44.1) in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z \tag{44.3}$$

and we refer to the surface as an **elliptic paraboloid** (Fig. 17). All sections parallel to the planes $x = 0$ and $y = 0$ are parabolae, whilst sections parallel to $z = 0$ are ellipses which are real for $z > 0$. No part of the surface lies below the plane $z = 0$. The elliptic paraboloid is a surface of revolution if $a = b$.

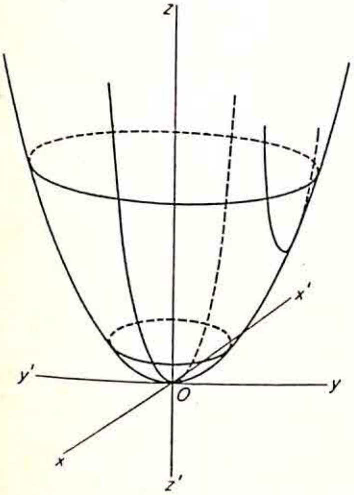

FIG. 17

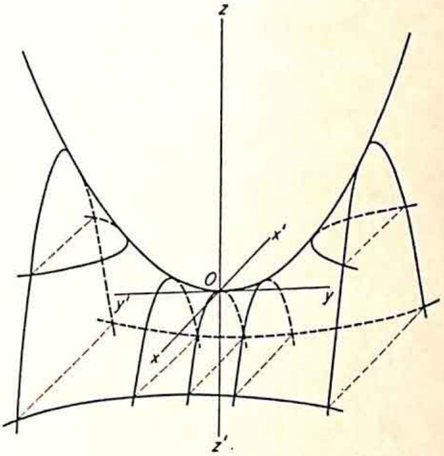

FIG. 18

Case 2. If a and b in (44.1) have opposite signs, we may rewrite it in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z \tag{44.4}$$

and we call the surface a **hyperbolic paraboloid** (Fig. 18). All sections parallel to $x = 0$ and $y = 0$ are parabolae, whilst sections parallel to $z = 0$ are hyperbolae. The section by the plane $z = 0$ itself is the line-pair

$$z = 0; \quad x/a + y/b = 0 \text{ and } z = 0; \quad x/a - y/b = 0.$$

The value of $x^2/a^2 - y^2/b^2 = 2z$ changes sign with z and so the directions of the conjugate and transverse axes of the hyperbolic sections parallel to $z = 0$ interchange on passing through $z = 0$.

A hyperbolic paraboloid is *never* a surface of revolution.

45. Polar properties

We can repeat the theory outlined in §37. With the notation

$$F_A \equiv ax_A{}^2 + by_A{}^2 + 2rz_A; \quad F_B \equiv ax_B{}^2 + by_B{}^2 + 2rz_B;$$
$$T_{AB} = T_{BA} \equiv ax_Ax_B + by_Ay_B + r(z_A + z_B);$$
$$T_A \equiv ax_Ax + by_Ay + r(z + z_A),$$

and the usual definitions, the results are as follows:

(i) The point P which divides AB in the ratio σ/τ lies on the paraboloid if

$$\tau^2F_A + 2\tau\sigma T_{AB} + \sigma^2F_B = 0. \tag{45.1}$$

(ii) The points A and B are conjugate if $T_{AB} = 0$.

(iii) The polar of A is the plane $T_A = 0$.

(iv) The pole of $\lambda x + \mu y + \nu z + \rho = 0$ is at the point

$$(\lambda r/a\nu, \quad \mu r/b\nu, \quad \rho/\nu).$$

(v) The two planes $\lambda_1x + \mu_1y + \nu_1z + \rho_1 = 0$ and $\lambda_2x + \mu_2y + \nu_2z + \rho_2 = 0$ are conjugate if

$$\lambda_1\lambda_2/a + \mu_1\mu_2/b + (\nu_1\rho_2 + \nu_2\rho_1)/r = 0.$$

(vi) The tangent plane at A is $T_A = 0$.

(vii) The plane $\lambda x + \mu y + \nu z + \rho = 0$ is a tangent plane if $\lambda^2/a + \mu^2/b + 2\nu\rho/r = 0$.

(viii) The enveloping cone from A is given by $F_AF - T_A{}^2 = 0$.

Example 1. Show that the plane $\lambda x + \mu y + \nu z = + r(\lambda^2/a + \mu^2/b)/2\nu$ is always a tangent plane of the paraboloid $ax^2 + by^2 + 2rz = 0$.

Example 2. Prove that the locus of the point of intersection of three mutually orthogonal tangent planes of the paraboloid $ax^2 + by^2 + 2rz = 0$ is the plane $2z = r(1/a + 1/b)$.

46. Generators

Joachimsthal's quadratic equation (45.1) is an identity if AB is a generator. The necessary and sufficient conditions are

$$F_A = 0; \quad F_B = 0; \quad T_{AB} = 0. \tag{46.1}$$

The parametric coordinates of any point on the line AB are given by (9.4). The tangent plane at this point has the equation

$$\eta(x, y, z) \equiv a\{(1-t)x_A + tx_B\}x + b\{(1-t)y_A + ty_B\}y \\ + r\{z + (1-t)z_A + tz_B\} = 0.$$

We readily verify that

$$\eta\{(1-s)x_A + sx_B,\quad (1-s)y_A + sy_B,\quad (1-s)z_A + sz_B\} \equiv 0$$

in virtue of (46.1). Hence a generator lies completely in the tangent plane at any point on it. The section of the paraboloid by the tangent plane must now be a line-pair and so in general two generators pass through a point and they determine the tangent plane at that point.

From equations (46.1) we have

$$(ax_A{}^2 + by_A{}^2)(ax_B{}^2 + by_B{}^2) - (ax_Ax_B + by_Ay_B)^2 \\ = (-2rz_A)(-2rz_B) - r^2(z_A + z_B)^2.$$

That is, $$ab(x_Ay_B - x_By_A)^2 = -r^2(z_A - z_B)^2.$$

Thus the generators are only real if a and b have opposite signs. Hence the hyperbolic paraboloid has real generators.

47. Generators of hyperbolic paraboloid

The equation (44.4) of the hyperbolic paraboloid may be written $(x/a + y/b)(x/a - y/b) = 2z$. Hence the generators are the lines of intersection of the planes

$$\frac{x}{a} + \frac{y}{b} = \frac{z}{\lambda}; \quad \frac{x}{a} - \frac{y}{b} = 2\lambda, \tag{47.1}$$

or the planes

$$\frac{x}{a} + \frac{y}{b} = 2\mu; \quad \frac{x}{a} - \frac{y}{b} = \frac{z}{\mu}. \tag{47.2}$$

The generators (47.1) and (47.2) are called the λ-system and the μ-system respectively.

The reader is asked to verify (compare §39) that no two generators of the same system intersect but that each generator of the λ-system intersects every generator of the μ-system at the point

$$(a(\lambda + \mu), \quad b(\mu - \lambda), \quad 2\lambda\mu).$$

Example 3. Show that the locus of the points of intersection of orthogonal generators of the hyperbolic paraboloid $x^2/a^2 - y^2/b^2 - 2z = 0$ is the plane $z = (b^2 - a^2)/2$. (Compare Example 2.)

48. Normals

The normal at A has the equations

$$\frac{x - x_A}{ax_A} = \frac{y - y_A}{by_A} = \frac{z - z_A}{r}. \tag{48.1}$$

Again we ask: "How many normals pass through the fixed point K?" We have

$$\frac{x_K - x_A}{ax_A} = \frac{y_K - y_A}{by_A} = \frac{z_K - z_A}{r} = t \text{ (say)}. \tag{48.2}$$

Therefore,

$$x_A = x_K/(1 + at); \quad y_A = y_K/(1 + bt); \quad z_A = z_K - rt. \tag{48.3}$$

By substitution in $F_A = 0$, we obtain

$$\frac{ax_K^2}{(1 + at)^2} + \frac{by_K^2}{(1 + bt)^2} + 2r(z_K - rt) = 0. \tag{48.4}$$

This equation is of the fifth degree in t. Hence, in general, five normals can be drawn from a given point to a paraboloid.

If the paraboloid is of revolution ($a = b$) or is a paraboloid cylinder ($a = 0$), equation (48.4) reduces to a cubic and in that case only three normals can be drawn through a given point.

We find the equation of the cone passing through these five normals by eliminating x_A, y_A, z_A and t from the equations

$$(x - x_K)/ax_A = (y - y_K)/by_A = (z - z_K)/r$$

of the normal KA and equations (48.3). The result is

$$ab(z - z_K)(x_K y - y_K x) + r(a - b)(x - x_K)(y - y_K) = 0. \tag{48.5}$$

Example 4. Calculate the shortest distance between the normals to $ax^2 + by^2 + 2rz = 0$ at the points $(2r, 0, -2ar)$ and $(0, 2r, -2br)$.

49. Centre of a section

The straight line through A in the direction $\{l, m, n\}$ has the equations (9.1). Its points of intersection with the paraboloid (44.1) are determined by the roots of the quadratic equation (44.2). Hence A is the centre of the chord through it in the direction $\{l, m, n\}$ if (44.2) has equal but opposite roots in t. That is

$$alx_A + bmy_A + rn = 0. \tag{49.1}$$

Accordingly, the equation of the plane whose centre is at A is obtained by eliminating l, m, n from (9.1) and (41.1). The result is

$$ax_A(x - x_A) + by_A(y - y_A) + r(z - z_A) = 0.$$

If A is the centre of the section by the plane $\lambda x + \mu y + \nu z + \rho = 0$, we obtain on comparing coefficients

$$ax_A/\lambda = by_A/\mu = r/\nu = -(ax_A{}^2 + by_A{}^2 + cz_A{}^2)/\rho.$$

The solution of these equations yields

$$A \equiv \left(\frac{\lambda r}{a\nu},\ \frac{\mu r}{b\nu},\ -\frac{\rho}{\nu} - \frac{r}{\nu^2}\left(\frac{\lambda^2}{a} + \frac{\mu^2}{b}\right)\right). \tag{49.2}$$

Example 5. Obtain the locus of the centres of sections of the paraboloid $ax^2 + by^2 + 2rz = 0$ which are at a constant distance p from the vertex.

50. Axes of plane section

In order to follow this section, the reader is advised to read §73 on the translation of axes. (Compare §43.) We wish to investigate the axes of the section of the paraboloid (44.1) by the plane $\lambda x + \mu y + \nu z + \rho = 0$. Translate the axes to the centre of the section. By (49.2) the equations of the translation are

$$x = X + \lambda r/a\nu; \quad y = Y + \mu r/b\nu;$$
$$z = Z - \rho/\nu - r(\lambda^2/a + \mu^2/b)/\nu^2.$$

The equations of the plane and paraboloid become $\lambda X + \mu Y + \nu Z = 0$ and

$$a\left(X + \frac{\lambda r}{a\nu}\right)^2 + b\left(Y + \frac{\mu r}{b\nu}\right)^2 + 2r\left[Z - \frac{\rho}{\nu} - \frac{r}{\nu^2}\left(\frac{\lambda^2}{a} + \frac{\mu^2}{b}\right)\right] = 0.$$

In virtue of $\lambda X + \mu Y + \nu Z = 0$, this equation reduces to

$$aX^2 + bY^2 + \omega = 0, \tag{50.1}$$

where

$$\omega = -r^2[\lambda^2/a + \mu^2/b + 2\nu\rho/r]/\nu^2. \tag{50.2}$$

The section we require is thus the same as the section of the cylinder (50.1) by the plane $\lambda X + \mu Y + \nu Z = 0$. This is a particular case of §42 corresponding to $c = 0$ and $\omega = d$. Hence the squares of the lengths of the semi-axes are the roots in R^2 of the quadratic equation

$$\frac{\lambda^2}{aR^2 + \omega} + \frac{\mu^2}{bR^2 + \omega} + \frac{\nu^2}{\omega} = 0, \tag{50.3}$$

and the corresponding direction-ratios are

$$\{\lambda/(aR^2 + \omega), \quad \mu/(bR^2 + \omega), \quad \nu/\omega\}.$$

Example 6. Calculate the principal axes of a plane section through the vertex of the paraboloid of revolution $x^2 + y^2 = 2rz$ in terms of r and the angle α between the plane section and the axis of the paraboloid.

Example 7. Show that any plane parallel to the z-axis intersects the paraboloid $ax^2 + by^2 + 2rz = 0$ in a parabola.

Miscellaneous examples

1. Show that the perpendiculars from the origin to the generators of the paraboloid $x^2/a^2 - y^2/b^2 = 2z$ generate the cones $(x/a \pm y/b)(ax \pm by) + 2z^2 = 0$.

2. Show that all the planes $\sqrt{a - b}x - \sqrt{b}z + \rho = 0$ $(a > b > 0, \rho > 0)$ cut the paraboloid $ax^2 + by^2 = 2z$ in circular sections.

3. Show that the locus of the points from which three mutually orthogonal tangent lines can be drawn to the paraboloid $ax^2 + by^2 + 2rz = 0$ is the paraboloid of revolution $ab(x^2 + y^2) + 2r(a + b)z - r^2 = 0$.

4. Prove that the plane $\lambda x + \mu y + \nu z + \rho = 0$ cuts the hyperboloid $ax^2 + by^2 + 2rz = 0$ in a rectangular hyperbola if $b\lambda^2 + a\mu^2 + (a + b)\nu^2 = 0$.

5. Find the locus of the centres of the sections of the paraboloid $ax^2 + by^2 + 2rz = 0$ which are of constant area πc^2.

6. Obtain the condition that the straight line $(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n$ be a tangent line to the paraboloid $ax^2 + by^2 + 2rz = 0$.

Hence calculate the angle between the two tangent lines parallel to the z-plane through (1, 2, 3) to the paraboloid $x^2 + 2y^2 = 2z$.

Answers

4. $2r(a - b)(1 - 2ab)/(a^2 + b^2 + 4a^2b^2)^{\frac{1}{2}}$. **5.** $(ax^2 + by^2 + rz)^2 = p^2(a^2x^2 + b^2y^2 + r^2)$. **6.** $2r \cot \alpha$, $2r \cot \alpha \operatorname{cosec} \alpha$.

Miscellaneous examples. 5. $(a^2x^2 + b^2y^2 + r^2)(ax^2 + by^2 + 2rz)^2 = abc^4r^2$. **6.** $(a\alpha l + b\beta m + rn)^2 = (al^2 + bm^2)(a\alpha^2 + b\beta^2 + 2r\gamma)$, $\cos^{-1} 2\sqrt{13}/13$.

CHAPTER VII

HOMOGENEOUS COORDINATES

51. Homogeneous coordinates

For many purposes it is useful to introduce four quantities X, Y, Z, W not all zero defined by

$$X:Y:Z:W = x:y:z:1.$$

The quantities $(X,\ Y,\ Z,\ W)$ are called **homogeneous Cartesian coordinates** of the point (x, y, z). Since $\rho X:\rho Y:\rho Z:\rho W = X:Y:Z:W$ we can also select $(\rho X, \rho Y, \rho Z, \rho W)$ as the homogeneous coordinates of the point $(x,\ y,\ z)$.

It will be convenient to use x, y, z, w instead of X, Y, Z, W as homogeneous coordinates. If we require at any stage to revert to non-homogeneous coordinates, it is only necessary to put $w = 1$.

52. Point at infinity

The parametric equation of the straight line through A with *direction-cosines* $[l,\ m,\ n]$ are $x = x_A + lt$, $y = y_A + mt$, $z = z_A + nt$. The homogeneous coordinates of any point P on the straight line are $P \equiv (x_A + lt,\ \ y_A + mt,\ \ z_A + nt,\ \ 1)$, or equally well $P \equiv (x_A/t + l,\ y_A/t + m,\ z_A/t + n,\ 1/t)$ obtained on division by t. The distance PA is t and as we allow t to tend to infinity the point P tends to an infinite distance from A. In the limit as t tends to infinity, the coordinates of P become $(l, m, n, 0)$ and we say that these are the coordinates of the **point at infinity** on the straight line. Clearly all parallel lines have the same point at infinity.

53. Straight line

With homogeneous coordinates the equation (9.3) of the straight line AB becomes

$$\frac{w_A x - w x_A}{w_A x_B - w_B x_A} = \frac{w_A y - w y_A}{w_A y_B - w_B y_A} = \frac{w_A z - w z_A}{w_A z_B - w_B z_A} = \frac{\sigma}{\rho}\ \text{(say)}.$$

Thus

$$\rho w_A x = (\rho w - \sigma w_B)x_A + \sigma w_A x_B, \text{ etc.}$$

Let us introduce τ by $\rho w - \sigma w_B = \tau w_A$. Then the homogeneous parametric equations of AB are

$$\left.\begin{aligned} \rho x &= \tau x_A + \sigma x_B, \\ \rho y &= \tau y_A + \sigma y_B, \\ \rho z &= \tau z_A + \sigma z_B, \\ \rho w &= \tau w_A + \sigma w_B. \end{aligned}\right\} \tag{53.1}$$

If B is the point at infinity $(l, m, n, 0)$, these equations represent the straight line through A in the direction $\{l, m, n\}$.

Consider three collinear points A, B and C. We may substitute the coordinates of C in (53.1) and eliminate the ratios $\rho:\tau:\sigma$ from every set of three of the four equations. The result is that the 3×4 matrix† formed from the homogeneous coordinates of A, B and C is of rank two. Hence, the necessary and sufficient condition that the three points A, B, C be collinear is that

$$R\begin{pmatrix} x_A & y_A & z_A & w_A \\ x_B & y_B & z_B & w_B \\ x_C & y_C & z_C & w_C \end{pmatrix} = 2.$$

Only two independent conditions are involved in this result.

If the rank of this matrix is one, the rows are proportional and so the points A, B and C must coincide.

The coordinates of the point P which divides AB in the ratio $\lambda:\mu$ are given by

$$P \equiv \left(\frac{\mu x_A + \lambda x_B}{\mu + \lambda}, \ \frac{\mu y_A + \lambda y_B}{\mu + \lambda}, \ \frac{\mu z_A + \lambda z_B}{\mu + \lambda}\right).$$

If we change to homogeneous coordinates, we have

$$P \equiv (\mu x_A w_B + \lambda x_B w_A, \ \mu y_A w_B + \lambda y_B w_A, \ \mu z_A w_B + \lambda z_B w_A, \\ (\mu + \lambda) w_A w_B).$$

Make the substitution $\lambda w_A = \sigma$ and $\mu w_B = \tau$ and the result is

$$P \equiv (\tau x_A + \sigma x_B, \ \tau y_A + \sigma y_B, \ \tau z_A + \sigma z_B, \ \tau w_A + \sigma w_B). \tag{53.2}$$

---

† The reader unacquainted with matrix theory is recommended to read the Appendix on pp. 115, 116.

The ratio $\sigma/\tau = (w_A/w_B)(\lambda/\mu)$, thus $\sigma/\tau = \lambda/\mu = AP/PB$ if and only if $w_A = w_B$. We, therefore, in applications of (53.2) make the convention that $w_A = w_B$ for finite points A and B in order to retain $AP/PB = \sigma/\tau$.

Example 1. Check for collinearity the sets of points (*a*), $(1, -2, -1)$, $(-2, -4, -2)$, $(13, 6, 3)$; (*b*) $(1, t, t^2)$, $(t^2, 1, t)$, $(t, t^2, 1)$ for real t.

54. Plane

A plane can be determined by a point and two directions through that point. These directions determine a pencil of lines which in turn define a **line at infinity** in the plane. This line at infinity will be common to all parallel planes.

The fourth coordinate w of a point at infinity is always zero. In order to preserve the result that all linear equations represent planes, we agree to say that $w = 0$ represents the **plane at infinity.**

In homogeneous coordinates we may represent a plane by the equation

$$\lambda x + \mu y + \nu z + \rho w = 0. \tag{54.1}$$

The line at infinity in this plane is $\lambda x + \mu y + \nu z = w = 0$. (We do not attach any meaning to the direction-ratios of such a line.)

If the four points A, B, C, D lie on the plane (54.1), their coordinates satisfy this equation. The elimination of the ratios $\lambda:\mu:\nu:\rho$ yields the result that

$$R\begin{pmatrix} x_A & y_A & z_A & w_A \\ x_B & y_B & z_B & w_B \\ x_C & y_C & z_C & w_C \\ x_D & y_D & z_D & w_D \end{pmatrix} = 3$$

is the necessary and sufficient condition that A, B, C and D be coplanar. If the rank of this matrix is two, it follows from the preceding section that A, B, C and D are collinear, whilst the points are coincident if the rank is unity.

Example 2. Show that the straight lines through A and B in the directions $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$ respectively intersect if

$$\begin{vmatrix} x_A & y_A & z_A & w_A \\ x_B & y_B & z_B & w_B \\ l_1 & m_1 & n_1 & 0 \\ l_2 & m_2 & n_2 & 0 \end{vmatrix} = 0.$$

55. Intersection of three planes

We consider in this section the intersections of the three planes

$$\left.\begin{aligned} u_1 &\equiv \lambda_1 x + \mu_1 y + \nu_1 z + \rho_1 w = 0,\\ u_2 &\equiv \lambda_2 x + \mu_2 y + \nu_2 z + \rho_2 w = 0,\\ u_3 &\equiv \lambda_3 x + \mu_3 y + \nu_3 z + \rho_3 w = 0.\end{aligned}\right\} \tag{55.1}$$

By Cramer's rule, the solution of this set of equations is

$$\frac{x}{\begin{vmatrix} \mu_1 & \nu_1 & \rho_1\\ \mu_2 & \nu_2 & \rho_2\\ \mu_3 & \nu_3 & \rho_3\end{vmatrix}} = \frac{-y}{\begin{vmatrix} \lambda_1 & \nu_1 & \rho_1\\ \lambda_2 & \nu_2 & \rho_2\\ \lambda_3 & \nu_3 & \rho_3\end{vmatrix}} = \frac{z}{\begin{vmatrix} \lambda_1 & \mu_1 & \rho_1\\ \lambda_2 & \mu_2 & \rho_2\\ \lambda_3 & \mu_3 & \rho_3\end{vmatrix}} = \frac{-w}{\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1\\ \lambda_2 & \mu_2 & \nu_2\\ \lambda_3 & \mu_3 & \nu_3\end{vmatrix}}. \tag{55.2}$$

We introduce the two matrices

$$U \equiv \begin{pmatrix} \lambda_1 & \mu_1 & \nu_1 & \rho_1\\ \lambda_2 & \mu_2 & \nu_2 & \rho_2\\ \lambda_3 & \mu_3 & \nu_3 & \rho_3\end{pmatrix}; \qquad V \equiv \begin{pmatrix} \lambda_1 & \mu_1 & \nu_1\\ \lambda_2 & \mu_2 & \nu_2\\ \lambda_3 & \mu_3 & \nu_3\end{pmatrix}.$$

Since V is a sub-matrix of U it follows that $R(V) \leq R(U)$. Further, if $R(V) = 1$ the columns of V are proportional. Thus all the 3×3 determinants in U are zero. That is, $R(U) \leq 2$ if $R(V) = 1$.

Case I. Let $R(U) = 3$. If $R(V) = 3$, we see from (55.2) that the three planes intersect in a point whose fourth coordinate is not zero and so the point is finite. If $R(V) = 2$ this common point is at infinity. Hence the planes are all parallel to a line.

Case II. Let $R(U) = 2$. We verify that

$$u_1 \begin{vmatrix} \nu_2 & \rho_2\\ \nu_3 & \rho_3\end{vmatrix} + u_2 \begin{vmatrix} \nu_3 & \rho_3\\ \nu_1 & \rho_1\end{vmatrix} + u_3 \begin{vmatrix} \nu_1 & \rho_1\\ \nu_2 & \rho_2\end{vmatrix}$$
$$\equiv x \begin{vmatrix} \lambda_1 & \nu_1 & \rho_1\\ \lambda_2 & \nu_2 & \rho_2\\ \lambda_3 & \nu_3 & \rho_3\end{vmatrix} + y \begin{vmatrix} \mu_1 & \nu_1 & \rho_1\\ \mu_2 & \nu_2 & \rho_2\\ \mu_3 & \nu_3 & \rho_3\end{vmatrix} = 0.$$

Accordingly the three planes $u_i = 0$ for $i = 1, 2$ and 3 form a pencil of planes. The points at infinity in the directions of the normals to the three planes are $(\lambda_1, \mu_1, \nu_1, 0)$, $(\lambda_2, \mu_2, \nu_2, 0)$ and $(\lambda_3, \mu_3, \nu_3, 0)$.

Consequently by §53 these three points coincide if $R(V) = 1$. Then the normals are parallel and the three planes are parallel to one another. However, if $R(V) = 2$ the three points at infinity just mentioned are distinct and so the three planes intersect in a finite line.

Case III. Let $R(U) = 1$. We verify that

$$\rho_2 u_1 - \rho_1 u_2 \equiv \begin{vmatrix} \lambda_1 & \rho_1 \\ \lambda_2 & \rho_2 \end{vmatrix} x + \begin{vmatrix} \mu_1 & \rho_1 \\ \mu_2 & \rho_2 \end{vmatrix} y + \begin{vmatrix} \nu_1 & \rho_1 \\ \nu_2 & \rho_2 \end{vmatrix} z = 0.$$

That is, $u_1 = 0$ and $u_2 = 0$ are the same plane. Similarly $u_3 = 0$ also gives the same plane. The three planes are accordingly coincident.

These results are conveniently summarized in the following table:

RANK OF MATRIX V	RANK OF MATRIX U		
	3	2	1
3	Planes have a unique finite point of intersection	╳	╳
2	Planes parallel to a line (planes form a prism)	Pencil of planes with a finite axis	╳
1	╳	Parallel planes	Coincident planes

Example 3. Investigate the nature of the intersection of the following sets of three planes:

(*a*) $x - y + z = 1$, $2x - 2y + 2z = 3$, $x + 5y - 6z = 2$;
(*b*) $x + y + z = 1$, $9x - y - 3z = 5$, $x + 2y + 2z = 1$;
(c) $9x - 2y - 3z = -4$, $3x - 10y + 6z = 1$, $6x + 8y - 9z = -5$;
(*d*) $ax + by + cz + d = 0$, $bx + cy + az + d = 0$, $cx + ay + bz + d = 0$ subject to (i) $a + b + c = 0$, (ii) $bc + ca + ab - b^2 - c^2 - a^2 = 0$. $(a, b, c \neq .)$

56. Circle at infinity

Consider the sphere whose equation is given by (18.2). On the introduction of homogeneous coordinates this equation becomes

$$a(x^2 + y^2 + z^2) + 2pxw + 2qyw + 2rzw + dw^2 = 0. \quad (56.1)$$

We now ask: "Has the sphere any points at infinity?" That is, are there any points on the sphere whose fourth homogeneous coordinate is zero? The answer is provided by the solution of equation (56.1) together with $w = 0$. These two equations simplify to

$$x^2 + y^2 + z^2 = w = 0. \tag{56.2}$$

Thus all spheres cut the plane at infinity in the same curve with equations (56.2). Since all plane sections of a sphere are circles, we call this curve the **circle at infinity** or **absolute** and it is generally denoted by Ω.

Conversely, if the general surface of the second degree

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2pxw \\ + 2qyw + 2rzw + dw^2 = 0$$

contains Ω, then it is satisfied identically by equations (56.2). Hence $f = g = h = 0$ and $a = b = c$. Thus the surface is a sphere.

The straight line at infinity in a plane cuts any circle of that plane in the two circular points of the plane. A plane cuts the circle at infinity in the two points of intersection of the line at infinity in the plane and Ω. Thus a plane cuts Ω in the two circular points lying in the plane.

A line which intersects the circle at infinity is called an **isotropic line.** The line through A in the direction $\{l, m, n\}$ is isotropic if the equations $(w_A x - x_A w)/l = (w_A y - y_A w)/m = (w_A z - z_A w)/n$ of the straight line written in homogeneous coordinates and the equations $x^2 + y^2 + z^2 = w = 0$ of the circle at infinity are consistent. That is, if $l^2 + m^2 + n^2 = 0$.

All isotropic lines through a point form a cone, called an **isotropic cone.** To obtain the equation of the isotropic cone through A, we eliminate l, m, n from $(x - x_A)/l = (y - y_A)/m = (z - z_A)/n$ and $l^2 + m^2 + n^2 = 0$. The result is $(x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2 = 0$. Accordingly the isotropic cone is identical with the point sphere at its vertex.

A plane which touches the circle at infinity is called an **isotropic plane.** The plane $\lambda x + \mu y + \nu z + \rho w = 0$ is isotropic if the equations $\lambda x + \mu y + \nu z = 0$ and $x^2 + y^2 + z^2 = 0$ have coincident solutions. The reader is asked to verify that the condition is $\lambda^2 + \mu^2 + \nu^2 = 0$.

By retaining a in equation (56.1) we may ask for the interpretation corresponding to $a = 0$. The sphere then degenerates to

$$w(2px + 2qy + 2rz + dw) = 0.$$

That is, a plane together with the plane at infinity must be regarded as a special case of a sphere. Of course, the centre at $(-p/a, -q/a, -r/a)$ and the radius $\sqrt{p^2 + q^2 + r^2 - ad}/a$ all tend to infinity.

57. Circular sections of central quadric

In homogeneous coordinates the central quadric (33.1) has the equation $ax^2 + by^2 + cz^2 + dw^2 = 0$. Its section by the plane at infinity is the conic $ax^2 + by^2 + cz^2 = w = 0$. This conic C cuts the circle Ω at infinity $x^2 + y^2 + z^2 = w = 0$ in four points P, Q, R, S (Fig. 19). P and Q are the circular points at infinity for any plane through PQ. This plane cuts the central quadric in a conic which passes through the circular points. Thus this conic is a circle and so PQ determines a pencil of parallel planes which cut the quadric in circular sections. Similarly there are five other such pencils of parallel planes corresponding to PR, PS, QR, QS and RS. These six lines are the degenerate members of the pencil of conics $ax^2 + by^2 + cz^2 - \lambda(x^2 + y^2 + z^2) = w = 0$. The lines correspond to $\lambda = a$, $\lambda = b$ and $\lambda = c$. Consequently the six pencils of parallel planes of circular section are the parallels to

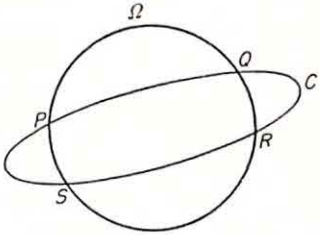

FIG. 19

$$(a - b)y^2 - (c - a)z^2 = 0, \tag{57.1}$$

$$(b - c)z^2 - (a - b)x^2 = 0, \tag{57.2}$$

$$(c - a)x^2 - (b - c)y^2 = 0. \tag{57.3}$$

We have $(a - b) + (b - c) + (c - a) = 0$. If the central quadric is not a surface of revolution, it follows that either one of $a - b$,

$b - c$, $c - a$ is positive and the other two negative or one is negative and the other two positive. Hence only one of the equations (57.1), (57.2) and (57.3) yields real planes. That is, a central quadric has only two pencils of plane circular sections.

In the case of a quadric of revolution, $b = c$ (say) and the planes (57.1)–(57.3) become respectively $y^2 + z^2 = 0$, $x^2 = 0$, $x^2 = 0$. That is, the points P, Q, R and S coincide in pairs. The planes $y^2 + z^2 = 0$ correspond to coincident circular points and so do not give circular sections. Consequently, there is only one pencil of parallel planes of circular section of a central quadric of revolution. Furthermore, its section with the plane at infinity has double contact with the circle at infinity.

The points of contact of tangent planes parallel to the planes of circular section are called **umbilics**.

Example 4. Obtain the real parallel planes of circular section of the quadrics (a) $2x^2 + 3y^2 + 7z^2 = 4$, (b) $x^2 - 3y^2 - 8z^2 = 7$, (c) $x^2 + y^2 - 3z^2 = 1$.

58. Paraboloid

In homogeneous coordinates, the equation (44.1) of the paraboloid becomes $ax^2 + by^2 + 2rzw = 0$. The section by the plane at infinity is thus the line-pair $ax^2 + by^2 = w = 0$. Hence the plane at infinity cuts the paraboloid in two generators. Accordingly, by §46 the plane at infinity is a tangent plane to the paraboloid. The line-pair $ax^2 + by^2 = w = 0$ cuts Ω the circle at infinity $x^2 + y^2 + z^2 = w = 0$ in four points P, Q, R and S (Fig. 20). As in the previous

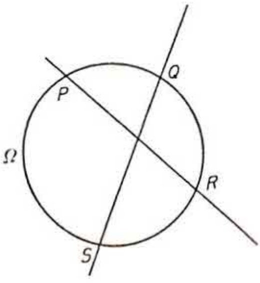

FIG. 20

section, the planes of circular section correspond to the degenerate members of the pencil of conics $ax^2 + by^2 - \lambda(x^2 + y^2 + z^2) = w = 0$. The degeneracies give $\lambda = 0$, $\lambda = a$, $\lambda = b$. The plane-pair

$ax^2 + by^2 = 0$ corresponding to $\lambda = 0$ sets up the pencils of parallel planes $\sqrt{a}x \pm \sqrt{-b}y =$ constant and these planes cut the paraboloid (44.1) in generators since the equation may be written $(\sqrt{a}x + \sqrt{-b}y)(\sqrt{a}x - \sqrt{-b}y) = -2rz$. Therefore $\lambda = 0$ does not yield circular sections (unless we include the degeneracy of a circle to a line together with the line at infinity). Thus there are four pencils of parallel planes of circular section parallel to

$$(b - a)y^2 - az^2 = 0, \tag{58.1}$$

$$(a - b)x^2 - bz^2 = 0. \tag{58.2}$$

We easily verify that the elliptic paraboloid for which $ab > 0$ possesses two real pencils of plane circular sections, whilst the hyperbolic paraboloid for which $ab < 0$ has no real planes of circular section.

For a paraboloid of revolution $a = b$ and so there is only one pencil of parallel planes of circular section, namely $z = 0$. The section of the paraboloid with the plane at infinity is a double generator and so this section has double contact with the circle at infinity.

Example 5. Obtain the real parallel planes of circular section of the paraboloids (*a*) $9x^2 + 25y^2 - 5z = 0$, (*b*) $z^2 + 2x^2 - 3y = 0$.

Miscellaneous examples

1. Obtain the necessary and sufficient conditions that the three planes $x + y + z + 1 = 0$, $\lambda x + \mu y + \nu z + \rho = 0$, $\lambda^3 x + \mu^3 y + \nu^3 z + \rho^3 = 0$ should (*a*) intersect in a unique finite point, (*b*) be parallel to a straight line, (*c*) form a pencil of planes with a finite axis, given that no two of λ, μ, ν, ρ are equal.

2. Calculate the coordinates of the real umbilics of the quadrics $ax^2 + by^2 + cz^2 + d = 0$ $(d < 0)$ given (*a*) $0 < a < b < c$, (*b*) $c < 0 < a < b$, (*c*) $b < c < 0 < a$.

3. Calculate the coordinates of the real umbilics of the paraboloid $ax^2 + by^2 + 2rz = 0$ given $0 < a < b$.

Answers

1. (*a*) Yes; (*b*) No. **3.** (*a*) All parallel to the straight line $6x - 7 = (6y - 1)/7 = z$; (*b*) Intersect at $(1, -2, 2)$; (*c*) pencil of planes through the straight line $(x - 1)/2 = (y - 2)/3 = (z - 3)/4$; (*d*) (i) planes form a prism, (ii) planes form a pencil with a finite axis. **4.** (*a*) $x \pm 2z + \rho = 0$, (*b*) $2x \pm \sqrt{5}z + \rho = 0$, (*c*) $z + \rho = 0$. **5.** (*a*) $4y \pm 3z + \rho = 0$, (*b*) $x \pm y + \rho = 0$.

Miscellaneous examples. 1. (*a*) $\lambda + \mu + \nu \neq 0$, (*b*) $\lambda + \mu + \nu = 0$, (*c*) impossible.

2.
$$(a)\ \left(\pm\sqrt{\frac{cd(b-a)}{ab(a-c)}},\quad 0,\quad \pm\sqrt{\frac{ad(c-b)}{bc(a-c)}}\right),$$

$$(b)\ \left(0,\quad \pm\sqrt{\frac{cd(b-a)}{ab(c-b)}},\quad \pm\sqrt{\frac{bd(a-c)}{ca(c-b)}}\right),$$

$$(c)\ \left(\pm\sqrt{\frac{bd(a-c)}{ac(b-a)}},\quad \pm\sqrt{\frac{ad(c-b)}{bc(b-a)}},\quad 0\right).$$

3. $$\left(0,\quad \pm\sqrt{\frac{b-a}{a}}\,\frac{r}{b},\quad \frac{(a-b)r}{2ab}\right).$$

CHAPTER VIII

GENERAL QUADRIC

59. Quadric

We define a quadric surface as the locus of points such that their coordinates satisfy the most general equation of the second degree, namely

$$F \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0. \quad (59.1)$$

In chapter X, we shall show that this equation represents either a central quadric or a paraboloid including their degeneracies.

As in §§33 or 44, a straight line is either a generator or intersects the quadric in two points which may be real and distinct, real and coincident or conjugate complex. Further, it follows that any plane section of a quadric is a conic which may degenerate to a line-pair.

There are nine effective constants in equation (59.1), namely the ratios of the ten quantities a, b, c, d, f, g, h, p, q and r. Thus, in general, one unique quadric passes through nine given points. If three of the nine points lie on a straight line, the line is a generator. Hence a quadric is uniquely determined by three *skew* generators.

A conic is determined uniquely by five of its points and so in general a unique quadric passes through a conic and four other points not in its plane.

If a quadric contains two conics, their planes intersect in a straight line which cuts the quadric in two points common to the two conics. That is, such conics have two points of intersection (real or coincident or conjugate complex) and the pair of conics is then equivalent to eight given points. Consequently an infinite number of quadrics pass through two conics which intersect in two points. Note that two conics with one point in common are uniquely determined by nine points, yet the only quadric containing the two conics is the plane-pair of the conics.

60. Cofactor notation

Introduce the two matrices

$$\mathbf{\Delta} \equiv \begin{pmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{pmatrix}; \qquad \mathbf{D} \equiv \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}.$$

Let A, B, C, D, F, G, H, P, Q, R denote the cofactors of a, b, c, d, f, g, h, p, q, r respectively in the determinant Δ, whilst $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ denote the cofactors of a, b, c, f, g, h, in the determinant D. Then there exist algebraic identities of the type

$$aA + hH + gG + pP = \Delta \tag{60.1}$$

obtained by multiplying the elements of a row of Δ by the corresponding cofactors, and

$$hA + bH + fG + qP = 0 \tag{60.2}$$

obtained by multiplying the elements of a row of Δ by the cofactors of the elements of a different row.

Further, we have identities† of the type

$$a\mathscr{A} + h\mathscr{H} + g\mathscr{G} = D; \quad h\mathscr{A} + b\mathscr{H} + f\mathscr{G} = 0. \tag{60.3}$$

Example 1. Show that $\Delta = dD - (\mathscr{A}p^2 + \mathscr{B}q^2 + \mathscr{C}r^2 + 2\mathscr{F}qr + 2\mathscr{G}rp + 2\mathscr{H}pq)$. Deduce that $R(\Delta) \leq 3$ if $R(D) = 1$.

61. Conjugate points

Consider the quadric with the *homogeneous* equation

$$\begin{aligned} F \equiv F(x, y, z, w) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx \\ + 2hxy + 2pxw + 2qyw + 2rzw + dw^2 = 0. \end{aligned} \tag{61.1}$$

For brevity, let us write

$$\left.\begin{aligned} F_A \equiv F(x_A, y_A, z_A, w_A); \quad F_B \equiv F(x_B, y_B, z_B, w_B), \\ T_{AB} = T_{BA} = \frac{1}{2}\left[x_B \frac{\partial F_A}{\partial x_A} + y_B \frac{\partial F_A}{\partial y_A} + z_B \frac{\partial F_A}{\partial z_A} + w_B \frac{\partial F_A}{\partial w_A}\right] \\ = \frac{1}{2}\left[x_A \frac{\partial F_B}{\partial x_B} + y_A \frac{\partial F_B}{\partial y_B} + z_A \frac{\partial F_B}{\partial z_B} + w_A \frac{\partial F_B}{\partial w_B}\right], \end{aligned}\right\} \tag{61.2}$$

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† These identities are listed in full in *Analytical Conics*, pp. 76–7.

$$\left.\begin{aligned}T_A \equiv \frac{1}{2}\left[x\frac{\partial F_A}{\partial x_A} + y\frac{\partial F_A}{\partial y_A} + z\frac{\partial F_A}{\partial z_A} + w\frac{\partial F_A}{\partial w_A}\right]&\\ = \frac{1}{2}\left[x_A\frac{\partial F}{\partial x} + y_A\frac{\partial F}{\partial y} + z_A\frac{\partial F}{\partial z} + w_A\frac{\partial F}{\partial w}\right].&\end{aligned}\right\}\tag{61.2}$$

The coordinates of the point P which divides AB in the ratio $\sigma:\tau$ are given by (53.2). This point lies on the quadric (61.1) if

$$F(\tau x_A + \sigma x_B,\ \tau y_A + \sigma y_B,\ \tau z_A + \sigma z_B,\ \tau w_A + \sigma w_B) = 0.$$

That is,

$$\tau^2 F_A + 2\tau\sigma T_{AB} + \sigma^2 F_B = 0. \tag{61.3}$$

With the usual definition of conjugacy (see §§19 and 37) we obtain that the necessary and sufficient condition that A and B are conjugate with respect to the quadric (61.1) is $T_{AB} = 0$.

Example 2. For what values of λ are the points $(1, \lambda, \lambda)$ and $(\lambda, \lambda, 1)$ conjugate with respect to the quadric $yz + zx + xy = 1$.

62. Pole and polar

Again, with the definitions in section 20, the polar plane of A has the equation $T_A = 0$.

We now obtain the pole K† of the plane $\lambda x + \mu y + \nu z + \rho w = 0$. This equation represents the same plane as $T_K = 0$. The comparison of coefficients yields

$$\frac{1}{2}\frac{\partial F_K}{\partial x_K} \equiv ax_K + hy_K + gz_K + pw_K = k\lambda,$$

$$\frac{1}{2}\frac{\partial F_K}{\partial y_K} \equiv hx_K + by_K + gz_K + qw_K = k\mu,$$

$$\frac{1}{2}\frac{\partial F_K}{\partial z_K} \equiv gx_K + fy_K + cz_K + rw_K = k\nu,$$

$$\frac{1}{2}\frac{\partial F_K}{\partial w_K} \equiv px_K + qy_K + rz_K + dw_K = k\rho,$$

where k is some factor of proportionality. In virtue of equations of the type (60.1) and (60.2) we verify that the solution of these equations is

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† We use K instead of A to avoid confusion with the cofactor A.

$$K \equiv (A\lambda + H\mu + G\nu + P\rho,\ H\lambda + B\mu + F\nu + Q\rho,\\ G\lambda + F\mu + C\nu + R\rho,\ P\lambda + Q\mu + R\nu + D\rho). \quad (62.1)$$

The pole of the plane $u_1 \equiv \lambda_1 x + \mu_1 y + \nu_1 z + \rho_1 w = 0$ is at the point given by (62.1) with the suffix 1 on λ, μ, ν and ρ. This point lies on $u_2 \equiv \lambda_2 x + \mu_2 y + \nu_2 z + \rho_2 w = 0$, if

$$A\lambda_1\lambda_2 + B\mu_1\lambda_2 + C\nu_1\nu_2 + D\rho_1\rho_2 + F(\mu_1\nu_2 + \mu_2\nu_1)\\ + G(\nu_1\lambda_2 + \nu_2\lambda_1) + H(\lambda_1\mu_2 + \lambda_2\mu_1) + P(\lambda_1\rho_2 + \lambda_2\rho_1)\\ + Q(\mu_1\rho_2 + \mu_2\rho_1) + R(\nu_1\rho_2 + \nu_2\rho_1) = 0. \quad (62.2)$$

This relation is symmetrical in the suffices 1 and 2. Hence the pole of $u_2 = 0$ lies on the plane $u_1 = 0$.

When two planes are so situated that each passes through the pole of the other, they are called **conjugate planes**. Equation (62.2) states the necessary and sufficient condition that the two planes $u_1 = 0$ and $u_2 = 0$ be conjugate.

Example 3. Obtain the coordinates of the pole of the plane $x + y + z = 1$ with respect to the quadric $yz + zx + xy = 1$.

Example 4. Show that the polar plane of any point with respect to a cone passes through its vertex.

63. Tangent plane

As in §21, we obtain that the tangent lines at a point A form a plane called the tangent plane with the equation $T_A = 0$. The pole of a tangent plane is at its points of contact and so a tangent plane is self-conjugate. Hence from (62.2) the necessary and sufficient condition that $\lambda x + \mu y + \nu z + \rho w = 0$ be a tangent plane to the quadric (61.1) is

$$A\lambda^2 + B\mu^2 + C\nu^2 + D\rho^2 + 2F\mu\nu + 2G\nu\lambda\\ + 2H\lambda\mu + 2P\lambda\rho + 2Q\mu\rho + 2R\nu\rho = 0.$$

Example 5. Show that the plane $kx + y + k^2z + 1 = 0$ touches the quadric $xy = kz$.

64. Enveloping cone

The line AB is a tangent to the quadric if equation (61.3) has coincident roots. That is, $F_A F_B - T_{AB}^2 = 0$. Thus the enveloping cone with vertex at A formed by all the tangent lines through A has the equation $F_A F - T_A{}^2 = 0$. If the vertex A is a point at infinity, we refer to the **enveloping cylinder.**

The polar plane of A intersects the quadric (61.1) in the conic given by $T_A = F = 0$. This conic lies on the cone $F_A F - T_A{}^2 = 0$. Therefore, the polar plane of A is the plane which passes through the points of contact of all the tangent lines through A.

Example 6. Find the locus of a point such that the enveloping cone from the point to the quadric $yz + zx + xy + 1 = 0$ is cut by the z-plane in a parabola.

Example 7. Obtain the equation of the enveloping cylinder to the quadric $x^2 + y^2 + 2zx = 1$ in the direction $\{1, -1, 0\}$.

65. Generators

If the straight line AB is a generator, then the equation (61.3) is an identity and so $F_A = F_B = T_{AB} = 0$. Clearly A lies in the tangent plane at B and B lies in the tangent plane at A. Thus a generator is the line of intersection of the tangent planes at any two points on it. Hence a generator lies completely in the tangent plane at any point on it. The tangent plane at A cuts the quadric in a conic which must in this case degenerate to a line-pair. Accordingly there is a second generator through A. That is, the tangent plane at a point intersects the quadric in two generators which may be real and distinct, real and coincident or complex conjugate.

The general equation of the quadric through the two given *skew* straight lines $u_1 = u_2 = 0$ and $u_3 = u_4 = 0$ where $u_i \equiv \lambda_i x + \mu_i y + \nu_i z + \rho_i$ is

$$\alpha u_1 u_3 + \beta u_1 u_4 + \gamma u_2 u_3 + \delta u_2 u_4 = 0, \tag{65.1}$$

since this equation is quadratic and is satisfied identically when $u_1 = u_2 = 0$ and when $u_3 = u_4 = 0$. Equation (65.1) still retains *three* effective constants in keeping with the fact that the quadric has been made to satisfy six conditions.

Two intersecting straight lines can be represented by $u_1 = u_2 = 0$ and $u_3 = k_1 u_1 + k_2 u_2 = 0$. The equation

$$\alpha u_1 u_3 + \beta u_1(k_1 u_1 + k_2 u_2) + \gamma u_2 u_3 \\ + \delta u_2(k_1 u_1 + k_2 u_2) + \varepsilon(k_1 u_1 + k_2 u_2) = 0 \tag{65.2}$$

is quadratic, vanishes identically when $u_1 = u_2 = 0$ and when $u_3 = k_1u_1 + k_2u_2 = 0$ and has *four* effective constants. Consequently it represents the quadric passing through the two intersecting straight lines.

Example 8. Obtain the equation of the quadric with the three generators

(*a*) $y = 0, z = 0;\ z = 1, x = -1;\ x = 2, y = -3;$

(*b*) $x + y + z - 1 = x + 2y - z - 2 = 0;\quad x - y + 2z - 3 = x + y - 3z - 4 = 0;\ 3x - 5y - 4z - 8 = 2x + 4y + z + 2 = 0.$

Example 9. Find the equation of the quadric with the two generators $x + y = y + z = 0$ and $x + y + z - 1 = 2x + 3y - z + 2 = 0$ such that its section by the plane $z = 0$ is the parabola $5y^2 = 16x$.

66. Polar of a line

As t varies, the point

$$(x_A + t(x_B - x_A),\ y_A + t(y_B - y_A),\ z_A + t(z_B - z_A),\ 1) \quad (66.1)$$

always lies on the straight line AB. The polar plane of this point has the equation

$$\{x_A + t(x_B - x_A)\}\frac{\partial F}{\partial x} + \{y_A + t(y_B - y_A)\}\frac{\partial F}{\partial y} + \{z_A + t(z_B - z_A)\}\frac{\partial F}{\partial z} + \frac{\partial F}{\partial w} = 0.$$

That is,

$$(1 - t)T_A + tT_B = 0. \quad (66.2)$$

Accordingly any point on the line $T_A = T_B = 0$ which is the axis of the pencil of planes (66.2) is conjugate to any point on the line AB. Two lines such that all points on one line are conjugate to all points on the other are called **mutual polars** and each is called the **polar line** of the other. The polar line is the line of intersection of the polar planes of any two points on the line.

If a line intersects its polar line, the point of intersection is conjugate to all the points on both lines. Consequently the polar plane of this point is the plane determined by the two lines. This point lies in the plane and so the plane is a tangent plane with its point of contact at this point of intersection of the two polar lines. Thus the lines are tangent lines to the quadric.

If a line coincides with its polar line, all its points must lie on the quadric. Hence the line must be a generator of the quadric.

Example 10. Find the locus of the straight lines through A which are orthogonal to their polar lines with respect to the quadric $ax^2 + by^2 + cz^2 + d = 0$.

67. Conjugate lines

Consider two straight lines AB and CD. The polar line of CD is given by $T_C = T_D = 0$. The line AB intersects this line if the point with coordinates (66.1) lies on $T_C = 0$ and $T_D = 0$ for some value of t. That is

$$\{x_A + t(x_B - x_A)\} \frac{\partial F_C}{\partial x_C} + \{y_A + t(y_B - y_A)\} \frac{\partial F_C}{\partial y_C} + \{z_A + t(z_B - z_A)\} \frac{\partial F_C}{\partial z_C} + \frac{\partial F_C}{\partial w_C} = 0,$$

together with the equation obtained on replacement of C by D. These equations simplify to $T_{AC} + t(T_{BC} - T_{AC}) = 0$ and $T_{AD} + t(T_{BD} - T_{AD}) = 0$. The elimination of t yields

$$T_{AC}T_{BD} = T_{AD}T_{BC}. \tag{67.1}$$

This equation is still valid if we interchange A and B with C and D respectively. Hence the polar line of AB intersects CD.

Two lines such that each intersects the polar of the other are called **conjugate lines.**

Example 11. If $ABCD$ is a tetrahedron for which AB is conjugate to CD and AC is conjugate to BD, show that AD is conjugate to BC. Such a tetrahedron is said to be **self-conjugate.** (A **self-polar** tetrahedron by definition is one in which the polar plane of each vertex is the opposite face.)

68. Conjugate diameters

The centre of a quadric is defined to be the pole of the plane at infinity. From (62.1) we see that the centre is at the point (P, Q, R, D) and is a finite point if $D \neq 0$. It is worthy of note that the coordinates of the centre satisfy the three equations

$$\partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0.$$

A line through the centre is called a **diameter** and a plane through the centre is called a **diametral plane.** The centre, if a finite point, bisects all diameters because the centre and the point at infinity on the diameter are conjugate points.

Let us consider the case $D \neq 0$. Any point on the diameter in the direction $\{l, m, n\}$ has the coordinates $(P/D + lt,\ Q/D + mt,\ R/D + nt)$ and its polar plane has the equation

$$\left(\frac{P}{D} + lt\right)(ax + hy + gz + p) + \left(\frac{Q}{D} + mt\right)(hx + by + fz + q) + \left(\frac{R}{D} + nt\right)(gx + fy + cz + r) + px + qy + rz + d = 0.$$

By equations of the type (60.1) and (60.2) this reduces to

$$\frac{1}{2}t\left(l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z}\right) + \frac{\Delta}{D} = 0.$$

That is, the polar planes form a pencil of planes parallel to the plane

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0. \tag{68.1}$$

Again, by (60.1) and (60.2) the coefficients of l, m and n are all zero at the centre (P, Q, R, D). Hence (68.1) represents a diametral plane which is said to be **conjugate** to the direction $\{l, m, n\}$.

In the diametral plane (68.1) choose any diameter. Its conjugate diametral planes will intersect (68.1) in another diameter. We say that the three diameters form a triple of conjugate diameters and the three planes determined by them a triple of conjugate diametral planes.

Consider the three diameters of the central quadric $ax^2 + by^2 + cz^2 + d = 0$ which have direction-ratios $\{l_1, m_1, n_1\}$, $\{l_2, m_2, n_2\}$ and $\{l_3, m_3, n_3\}$. The plane conjugate to $\{l_1, m_1, n_1\}$ is $al_1x + bm_1y + cn_1z = 0$. All the points on the diameter in the direction $\{l_2, m_2, n_2\}$ lie in this plane if $al_1l_2 + bm_1m_2 + cn_1n_2 = 0$. Hence the three directions $\{l_i, m_i, n_i\}$ for $i = 1, 2, 3$ determine a triple of conjugate diameters if

$$al_il_j + bm_im_j + cn_in_j = 0. \quad (i \neq j), \quad (i, j = 1, 2, 3). \tag{68.2}$$

Consider the mid-points of all chords of the quadric parallel to the direction $\{l, m, n\}$. These points are conjugate to the point at infinity $(l, m, n, 0)$ and so lie on the plane (68.1). That is, the locus of these mid-points is the diametral plane conjugate to the direction $\{l, m, n\}$.

Select any point A on a diameter of a quadric. Let this diameter cut the polar plane α of A at the point C. Then the polar plane of C

is parallel to the plane α and so the planes intersect in a line at infinity. Thus the polar of C with respect to the conic of intersection of the plane α and the quadric is the line at infinity in its own plane. Accordingly C is the centre of the section α. That is, the centres of all sections of a quadric which are parallel to a diametral plane lie on the conjugate diameter.

If $D = 0$, the centre is the point at infinity $(P, Q, R, 0)$ and so the diameters are all parallel to the direction $\{P, Q, R\}$. It is still true that the mid-points of all chords parallel to $\{l, m, n\}$ lie on the polar plane of $(l, m, n, 0)$. That is, the locus of mid-points is the plane (68.1). We now say that this plane is the diametral plane conjugate to the direction $\{l, m, n\}$.

The central quadrics (including the non-degenerate cones) have a unique finite centre, whilst the paraboloids have a unique centre at infinity. However, the elliptic and hyperbolic cylinders (including the non-parallel plane pairs) $ax^2 + by^2 + d = 0$ possess the finite line of centres $x = y = 0$. On the other hand, the parabolic cylinders $by^2 + 2px = 0$ have a line of centres $y = w = 0$ at infinity. Finally, a pair of parallel planes $ax^2 + d = 0$ has a plane of centres $x = 0$.

Example 12. Show that conjugate diameters are also conjugate lines.

Example 13. Prove that the sum of the squares of three conjugate semi-diameters of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $a^2 + b^2 + c^2$.

Example 14. Show that the volume of the parallelepiped whose edges are three conjugate semi-diameters of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is abc.

69. Principal diametral planes

By definition, a **principal diametral plane** of a *central* quadric is orthogonal to its conjugate diameter which is then called a **principal axis.** Hence equation (68.1) represents a principal diametral plane if

$$\frac{al + hm + gn}{l} = \frac{hl + bm + fn}{m} = \frac{gl + fm + cn}{n} = \lambda \text{ (say)}. \qquad (69.1)$$

That is,

$$\left.\begin{aligned} (a - \lambda)l + \quad hm + \quad gn &= 0, \\ hl + (b - \lambda)m + \quad fn &= 0, \\ gl + \quad fm + (c - \lambda)n &= 0. \end{aligned}\right\} \qquad (69.2)$$

The elimination of l, m and n yields the determinantal equation

$$D_\lambda \equiv \begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0. \tag{69.3}$$

That is,

$$\psi(\lambda) \equiv \lambda^3 - (a+b+c)\lambda^2 + (\mathscr{A}+\mathscr{B}+\mathscr{C})\lambda - D = 0. \tag{69.4}$$

This equation is called the **discriminating cubic** and has three roots λ_1, λ_2, λ_3 which correspond in general to three principal axes $\{l_1, m_1, n_1\}$, $\{l_2, m_2, n_2\}$ and $\{l_3, m_3, n_3\}$ respectively. From the identity

$$l_1(al_2 + hm_2 + gn_2) + m_1(hl_2 + bm_2 + fn_2) + n_1(gl_2 + fm_2 + cn_2)$$
$$\equiv l_2(al_1 + hm_1 + gn_1) + m_2(hl_1 + bm_1 + fn_1) + n_2(gl_1 + fm_1 + cn_1)$$

and (69.1) we obtain

$$\lambda_2(l_1l_2 + m_1m_2 + n_1n_2) = \lambda_1(l_1l_2 + m_1m_2 + n_1n_2).$$

Consequently $l_1l_2 + m_1m_2 + n_1n_2 = 0$ if $\lambda_1 \neq \lambda_2$. Hence principal axes are generally mutually orthogonal.

Example 15. Show that the coordinate axes are the principal axes of the quadric $ax^2 + by^2 + cz^2 + d = 0$.

Example 16. Obtain the principal diametral planes of the quadric $2\sqrt{3}(yz - zx) + 2xy = 1$.

70. Discriminating cubic

First we shall show that all the roots of the discriminating cubic are real. If a complex root λ exists, the corresponding values of $l:m:n$ cannot all be real. Let us write l^*, m^*, n^* for the complex conjugates of l, m and n respectively. From (69.1) we obtain that

$$\lambda(ll^* + mm^* + nn^*) \equiv all^* + bmm^* + cnn^*$$
$$+ f(mn^* + m^*n) + g(nl^* + n^*l) + h(lm^* + l^*m).$$

But ll^*, mm^*, nn^*, $mn^* + m^*n$, $nl^* + n^*l$, $lm^* + l^*m$ are all real and so λ is real. It follows that the corresponding ratios $l:m:n$ are real.

Next we determine the condition that the discriminating cubic has a repeated root. If λ is a double root of (69.4) it is also a root of the derived equation $\psi'(\lambda) \equiv 3\lambda^2 - 2(a + b + c)\lambda + (\mathscr{A} + \mathscr{B} + \mathscr{C}) = 0$, which may be written

$$\{(b - \lambda)(c - \lambda) - f^2\} + \{(c - \lambda)(a - \lambda) - g^2\} + \{(a - \lambda)(b - \lambda) - h^2\} \equiv \mathscr{A}_\lambda + \mathscr{B}_\lambda + \mathscr{C}_\lambda = 0,$$

if we use the notation introduced in the Appendix p. 116. The inequality (A.1) then indicates that $R(\mathbf{D}_\lambda) \leq 1$.

If λ is a triple root of (69.4) it is also a root of the second derived equation $\psi''(\lambda) \equiv 6\lambda - 2(a + b + c) = 0$. That is,

$$(a - \lambda) + (b - \lambda) + (c - \lambda) = 0.$$

The inequality (A.2) of the Appendix then indicates that $R(\mathbf{D}_\lambda) = 0$.

Summing up, the discriminating cubic has a double root if $R(\mathbf{D}_\lambda) = 1$ and a triple root if $R(\mathbf{D}_\lambda) = 0$. In the latter case $a = b = c$ and $f = g = h = 0$. That is, the quadric is a sphere. We shall see later in §80 that a double root of the discriminating cubic corresponds to a surface of revolution. A similar result for the cone has already been obtained in §28.

71. Circular sections

As in §57, the circular sections are the sections by the pencils of parallel planes determined by the degenerate members of the pencil of conics at infinity

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda(x^2 + y^2 + z^2) = w = 0. \qquad (71.1)$$

The degeneracies correspond to the roots of the discriminating cubic $D_\lambda = 0$.

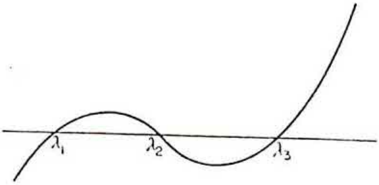

FIG. 21

The expression

$$(a - \lambda)x^2 + (b - \lambda)y^2 + (c - \lambda)z^2 + 2fyz + 2gzx + 2hxy$$

has real factors if $(b - \lambda)(c - \lambda) - f^2 < 0$, $(c - \lambda)(a - \lambda) - g^2 < 0$ and $(a - \lambda)(b - \lambda) - h^2 < 0$. By addition it follows that $\psi'(\lambda) < 0$.

The roots of $\psi(\lambda) = 0$ are real and $\psi(\lambda)$ tends to $+\infty$ as λ tends to $+\infty$ and $\psi(y)$ tends to $-\infty$ as λ tends to $-\infty$. Except for scale, the graph of $y = \psi(x)$ is thus depicted in Fig. 21. We see that $\psi'(\lambda) > 0$ at the smallest and largest root but $\psi'(\lambda) < 0$ at the middle root. Hence real circular sections, if they exist, correspond to the middle root of the discriminating cubic. These circular sections only fail to exist if the circles degenerate. (Compare §58 on the hyperbolic paraboloid when $\lambda = 0$ is the middle root.)

Example 17. Obtain the real parallel planes of circular section of the quadrics (*a*) $2\sqrt{3}(yz - zx) + 2xy = 1$; (*b*) $z^2 - xy = 1$.

72. Rank of matrix Δ

The polar planes of the four points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1) are respectively

$$\begin{aligned} \partial F/\partial x &\equiv ax + hy + gz + pw = 0, \\ \partial F/\partial y &\equiv hx + by + fz + qw = 0, \\ \partial F/\partial z &\equiv gx + fy + cz + rw = 0, \\ \partial F/\partial w &\equiv px + qy + rz + dw = 0. \end{aligned}$$

These four planes will form a tetrahedron if $R(\mathbf{\Delta}) = 4$.

If $R(\mathbf{\Delta}) = 3$, the four planes have a common point A whose coordinates satisfy the equations $\partial F_A/\partial x_A = \partial F_A/\partial y_A = \partial F_A/\partial z_A = \partial F_A/\partial w_A = 0$. Let B be any other point on the quadric. We can then set up Joachimsthal's quadratic equation (61.3). We have

$$2F_A \equiv x_A \partial F_A/\partial x_A + y_A \partial F_A/\partial y_A + z_A \partial F_A/\partial z_A + w_A \partial F_A/\partial w_A = 0$$

and so A lies on the quadric. Since B lies on the quadric we have $F_B = 0$. Further

$$2T_{AB} \equiv x_B \partial F_A/\partial x_A + y_B \partial F_A/\partial y_A + z_B \partial F_A/\partial z_A + w_B \partial F_A/\partial w_A = 0.$$

Hence (61.3) is an identity. That is, AB is a generator for every point B on the quadric. Thus the quadric consists of straight lines through A. Accordingly the quadric is a cone with vertex at A. (The cone becomes a cylinder if A is a point at infinity.)

If $R(\mathbf{\Delta}) = 2$, the four planes have a line in common. By the preceding argument all points of this line lie on the quadric and so the line joining any point on the quadric to any point on this line must be a generator. Consequently the quadric consists of two planes passing through this line.

Finally if $R(\Delta) = 1$, the four planes coincide and every point on the plane lies on the quadric. Further the line joining any point on the quadric to any point on this plane is a generator. That is, the quadric consists of two coincident planes.

Example 18. Examine the nature of the quadric $(x \cos \alpha - y \sin \alpha)^2 - (y \cos \alpha + z \sin \alpha)^2 + 2y - 1 = 0$ as α varies.

Miscellaneous examples

1. Obtain the equation of the common enveloping cone of the two quadrics $z^2 + 2xy + 2az = 0$ and $z^2 + 2xy + 2bz = 0$.

2. Find the locus of the poles of planes through the extremities of three conjugate semi-diameters of the quadric $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

3. Prove that the generators of the surface $fyz + gzx + hxy + fgh = 0$ through A are orthogonal if $x_A{}^2 + y_A{}^2 + z_A{}^2 = f^2 + g^2 + h^2$.

4. A variable quadric passes through the circle determined by the three fixed points $(a, 0, 0)$, $(0, a, 0)$ and $(0, 0, a)$. The quadric has the x-axis and y-axis as generators. Obtain the locus of the centre of the quadric.

5. Show that the two cylinders $41x^2 - 24xy + 34y^2 = 25$ and $25x^2 + 40xz + 34z^2 = 9$ have a common circular section. Find the equation of the plane of this circle and its radius.

6. If $-1 < f < 1$, show that the real planes of circular section of the quadric $2fyz + 2zx - 2xy = 1$ are parallel to the planes $fx + (1 \pm \sqrt{1 - f^2})(y - z) = 0$. What happens when $f = \pm 1$?

7. Show that the section of the quadric $yz + zx + xy = k$ by the plane $\lambda x + \mu y + \nu z + \rho = 0$ is a parabola if $\sqrt{\lambda} \pm \sqrt{\mu} \pm \sqrt{\nu} = 0$.

8. If three straight lines are mutually skew and all parallel to a plane, show that all their transversals are mutually skew and parallel to a plane.

9. Prove that the line of intersection of corresponding planes of two pencils of planes in homographic correspondence generate a quadric, which has the two axes as generators.

Answers

2. $-1, 1/3$. **6.** $z = \pm 1$. **7.** $x^2 + y^2 - z^2 + 2yz + 2zx + 2xy - 2 = 0$. **8.** (*a*) $3yz + 3zx + xy - 2y + 3z = 0$; (*b*) $9x^2 + 35y^2 - 35z^2 - 78yz - 22zx + 16xy - 35x - 21y - 77z - 14 = 0$. **9.** $5y^2 - 12z^2 + 5yz + 12zx - 16x + 16z = 0$. **10.** The cone $\Sigma bc(x - x_A)(y_A z - z_A y) = 0$. **16.** $x - y - \sqrt{6}z = 0$, $x + y = 0$ and $\sqrt{3}x - \sqrt{3}y + \sqrt{2}z = 0$. **17.** (*a*) $\sqrt{2}x - \sqrt{2}(2 \pm \sqrt{3})y + (\sqrt{3} \pm 1)z + k = 0$; (*b*) $x + y \pm z + k = 0$. **18.** Degenerates to a plane-pair if $\alpha = n\pi$ or to a cone (actually a cylinder) if $\alpha = (n + \frac{1}{2})\pi$.

Miscellaneous examples. **1.** $\{(a + b)z + 2ab\}^2 + 8abxy = 0$. **2.** $x^2/a^2 + y^2/b^2 + z^2/c^2 = 3$. **4.** Conic $x - y = 2x^2 + 2z^2 + 2zx - a(z + x) = 0$. **5.** $4x - 3y + 5z = 0, 1$. **6.** Surface is of revolution.

CHAPTER IX

TRANSFORMATION OF AXES—INVARIANTS

73. Translation of axes

Let P (Fig. 22) have coordinates (x, y, z) referred to the OX, OY, OZ coordinate system (called the *old* coordinate system), and coordinates $(\bar{x}, \bar{y}, \bar{z})$ referred to a *new* coordinate system $\bar{O}\bar{X}$, $\bar{O}\bar{Y}$, $\bar{O}\bar{Z}$ obtained by drawing respective parallels through $\bar{O}$ to OX, OY and OZ. We refer to this change or transformation of axes as a **translation** of axes.

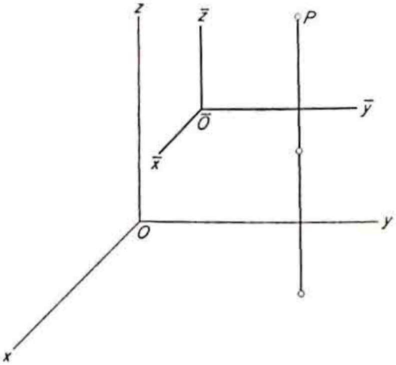

FIG. 22

Let the coordinates of the new origin $\bar{O}$ be (α, β, γ) referred to the old axes OX, OY, OZ. From the figure it is clear that

$$x = \bar{x} + \alpha; \quad y = \bar{y} + \beta; \quad z = \bar{z} + \gamma. \tag{73.1}$$

The inverse transformation is the translation back from the new axes to the old axes, and the corresponding equations are

$$\bar{x} = x - \alpha; \quad \bar{y} = y - \beta; \quad \bar{z} = z - \gamma.$$

If $f(x, y, z) = 0$ represents a surface with respect to the old axes, then $f(\bar{x} + \alpha, \bar{y} + \beta, \bar{z} + \gamma) = 0$ is the equation referred to the new axes. On the other hand, if $f(\bar{x}, \bar{y}, \bar{z}) = 0$ is the equation of the surface with respect to the new axes, then $f(x - \alpha, y - \beta, z - \gamma) = 0$ is the equation referred to the old axes.

74. Invariants of a quadratic form under translation

The quadratic form in x, y, z and $w = 1$ is

$$\begin{aligned} F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx \\ + 2hxy + 2px + 2qy + 2rz + d. \quad (74.1) \end{aligned}$$

We ask: "What functions of a, b, c, d, f, g, h, p, q, r are unaltered by a translation of axes?" These functions are called **invariants** of the quadratic form. Note carefully that invariants are not functions of x, y and z.

The translation (73.1) transforms the quadratic form (74.1) into

$$\begin{aligned} \bar{F}(\bar{x}, \bar{y}, \bar{z}) \equiv a(\bar{x} + \alpha)^2 + b(\bar{y} + \beta)^2 + c(\bar{z} + \gamma)^2 \\ + 2f(\bar{y} + \beta)(\bar{z} + \gamma) + zg(\bar{z} + \gamma)(\bar{x} + \alpha) \\ + 2h(\bar{x} + \alpha)(\bar{y} + \beta) + 2p(\bar{x} + \alpha) + 2q(\bar{y} + \beta) \\ + 2r(\bar{z} + \gamma) + d. \end{aligned}$$

Let us write

$$\begin{aligned} \bar{F}(\bar{x}, \bar{y}, \bar{z}) \equiv \bar{a}\bar{x}^2 + \bar{b}\bar{y}^2 + \bar{c}\bar{z}^2 + 2\bar{f}\bar{y}\bar{z} + 2\bar{g}\bar{z}\bar{x} \\ + 2\bar{h}\bar{x}\bar{y} + 2\bar{p}\bar{x} + 2\bar{q}\bar{y} + 2\bar{r}\bar{z} + \bar{d}. \end{aligned}$$

By equating coefficients, the equation connecting the old coefficients a, b, c, etc., and the new coefficients $\bar{a}$, $\bar{b}$, $\bar{c}$, etc. are

$$\begin{aligned} &\bar{a} = a; \quad \bar{b} = b; \quad \bar{c} = c; \quad \bar{f} = f; \quad \bar{g} = g; \quad \bar{h} = h; \\ &\bar{p} = a\alpha + h\beta + g\gamma + p; \quad \bar{q} = h\alpha + b\beta + f\gamma + q; \\ &\bar{r} = g\alpha + f\beta + c\gamma + r; \\ &\bar{d} = a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2p\alpha + 2q\beta \\ &\qquad + 2r\gamma + d, \\ &\quad = \bar{p}\alpha + \bar{q}\beta + \bar{r}\gamma + p\alpha + q\beta + r\gamma + d. \end{aligned}$$

The determinant

$$\bar{\Delta} \equiv \begin{vmatrix} \bar{a} & \bar{h} & \bar{g} & \bar{p} \\ \bar{h} & \bar{b} & \bar{f} & \bar{q} \\ \bar{g} & \bar{f} & \bar{c} & \bar{r} \\ \bar{p} & \bar{q} & \bar{r} & \bar{d} \end{vmatrix}$$

$$\equiv \begin{vmatrix} a & h & g & a\alpha + h\beta + g\gamma + p \\ h & b & f & h\alpha + b\beta + f\gamma + q \\ g & f & c & g\alpha + f\beta + c\gamma + r \\ a\alpha + h\beta + g\gamma + p & h\alpha + b\beta + c\gamma + r & g\alpha + f\beta + c\gamma + r & \bar{p}\alpha + \bar{q}\beta + \bar{r}\gamma + p\alpha + q\beta + r\gamma + d \end{vmatrix}.$$

Subtract α times the first column, β times the second column and γ times the third column from the fourth column. Next repeat the same process with the rows. The resulting determinant is Δ.

Accordingly we have the seven invariants a, b, c, f, g, h and Δ.

Example 1. Prove that the quadric $F = 0$ is a cone if $\Delta = 0$.

75. Rotation of axes

A change of axes in which the origin remains fixed is called a **rotation** of axes. Let the point P (Fig. 23) have coordinates (x, y, z) referred to the *old* axes OX, OY, OZ and coordinates $(\bar{x}, \bar{y}, \bar{z})$ with respect to a *new* coordinate system of three mutually perpendicular lines $O\bar{X}$, $O\bar{Y}$ and $O\bar{Z}$. Further, let the *direction-cosines* of the new axes with respect to the old axes be $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ respectively.

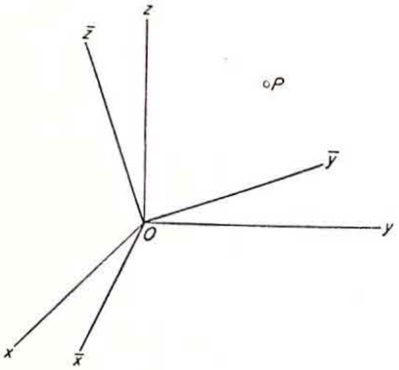

FIG. 23

Project OP on each of the old axes in turn and obtain in virtue of (7.1),

$$\left.\begin{aligned} x &= l_1\bar{x} + l_2\bar{y} + l_3\bar{z}, \\ y &= m_1\bar{x} + m_2\bar{y} + m_3\bar{z}, \\ z &= n_1\bar{x} + n_2\bar{y} + n_3\bar{z}. \end{aligned}\right\} \qquad (75.1)$$

We have

$$\left.\begin{aligned} &l_1^2 + m_1^2 + n_1^2 = 1; \quad l_2l_3 + m_2m_3 + n_2n_3 = 0; \\ &l_2^2 + m_2^2 + n_2^2 = 1; \quad l_3l_1 + m_3m_1 + n_3n_1 = 0; \\ &l_3^2 + m_3^2 + n_3^2 = 1; \quad l_1l_2 + m_1m_2 + n_1n_2 = 0. \end{aligned}\right\} \qquad (75.2)$$

By means of these equations, we solve (75.1) and obtain

$$\left.\begin{aligned} \bar{x} &= l_1x + m_1y + n_1z, \\ \bar{y} &= l_2x + m_2y + n_2z, \\ \bar{z} &= l_3x + m_3y + n_3z. \end{aligned}\right\} \tag{75.3}$$

These equations could have been written down immediately by noting that $[l_1, l_2, l_3]$, $[m_1, m_2, m_3]$ and $[n_1, n_2, n_3]$ are the direction-cosines of the old axes with respect to the new axes. Hence we also have

$$\begin{aligned} &l_1^2 + l_2^2 + l_3^2 = 1; \quad m_1n_1 + m_2n_2 + m_3n_3 = 0; \\ &m_1^2 + m_2^2 + m_3^2 = 1; \quad n_1l_1 + n_2l_2 + n_3l_3 = 0; \\ &n_1^2 + n_2^2 + n_3^2 = 1; \quad l_1m_1 + l_2m_2 + l_3m_3 = 0; \end{aligned}$$

We define the **modulus** of the rotation to be the determinant

$$M \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

We have

$$M^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + m_1m_2 + n_1n_2 & l_3l_1 + m_3m_1 + n_3n_1 \\ l_1l_2 + m_1m_2 + n_1n_2 & l_2^2 + m_2^2 + n_2^2 & l_2l_3 + m_2m_3 + n_2n_3 \\ l_3l_1 + m_3m_1 + n_3n_1 & l_2l_3 + m_2m_3 + n_2n_3 & l_3^2 + m_3^2 + n_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Thus $M = \pm 1$. For the identity rotation of the original right-handed system of axes into themselves, we have $l_1 = m_2 = n_3 = 1$ and $l_2 = l_3 = m_1 = m_3 = n_1 = n_2 = 0$ and so $M = +1$. A rotation to another right-handed system of axes can be performed continuously. Therefore M must vary continuously. Hence $M = +1$ for all rotations from a right-handed system of coordinates to any other right-handed system. $M = -1$ corresponds to a change from a right-handed system to a left-handed system or vice-versa.

Example 2. Prove that $l_1 = m_2n_3 - m_3n_2$; $m_1 = n_2l_3 - n_3l_2$; $n_1 = l_2m_3 - l_3m_2$.

Example 3. Show that the equations $\bar{x} = \frac{1}{3}(x - 2y + 2z)$, $\bar{y} = \frac{1}{3}(2x - y - 2z)$, $\bar{z} = \frac{1}{3}(2x + 2y + z)$ represent a rotation of axes. Hence show that $7x^2 + 6y^2 + 5z^2 + 4yz + 4xy - 3 = 0$ is the equation of an ellipsoid.

76. Invariants of quadratic form under rotation

The rotation (75.1) will transform the quadratic form (74.1) into

$$\bar F \equiv \bar a\bar x^2 + \bar b\bar y^2 + \bar c\bar z^2 + 2\bar f\bar y\bar z + 2\bar g\bar z\bar x \\ + 2\bar h\bar x\bar y + 2\bar p\bar x + 2\bar q\bar y + 2\bar r\bar z + \bar d.$$

For all rotations $x^2 + y^2 + z^2 = \bar x^2 + \bar y^2 + \bar z^2$ since each expression measures the square of the distance from the fixed origin. Thus for any given value of λ the quadratic form

$$U_\lambda \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda(x^2 + y^2 + z^2)$$

transforms into

$$\bar U_\lambda \equiv \bar a\bar x^2 + \bar b\bar y^2 + \bar c\bar z^2 + 2\bar f\bar y\bar z + 2\bar g\bar z\bar x + 2\bar h\bar x\bar y - \lambda(\bar x^2 + \bar y^2 + \bar z^2).$$

Now $U_\lambda = 0$ factorizes and then represents a plane-pair if

$$\begin{vmatrix} a - \lambda & h & g \\ h & b - \lambda & f \\ g & f & c - \lambda \end{vmatrix} = 0.$$

That is, λ is a root of the discriminating cubic

$$\lambda^3 - (a + b + c)\lambda^2 + (\mathscr{A} + \mathscr{B} + \mathscr{C})\lambda - D = 0. \qquad (76.1)$$

Similarly $\bar U_\lambda = 0$ factorizes and then represents a plane-pair if

$$\lambda^3 - (\bar a + \bar b + \bar c)\lambda^2 + (\bar{\mathscr{A}} + \bar{\mathscr{B}} + \bar{\mathscr{C}})\lambda - \bar D = 0. \qquad (76.2)$$

The equations $U_\lambda = 0$ and $\bar U_\lambda = 0$ represent the same surface with respect to different coordinate axes. It follows that if $U_\lambda = 0$ represents a plane-pair, $\bar U_\lambda = 0$ must represent a plane-pair for the *same* value of λ. Hence the cubic equations (76.1) and (76.2) are identical and so we have found the invariants

$$I \equiv a + b + c; \quad J \equiv \mathscr{A} + \mathscr{B} + \mathscr{C}; \quad D.$$

Next consider the quadratic form

$$F_\lambda \equiv F - \lambda(x^2 + y^2 + z^2 + 1)$$

which transforms under the rotation into

$$\bar F_\lambda \equiv \bar F - \lambda(\bar x^2 + \bar y^2 + \bar z^2 + 1).$$

By §72 the quadric $F_\lambda = 0$ represents a cone if

$$\begin{vmatrix} a-\lambda & h & g & p \\ h & b-\lambda & f & q \\ g & f & c-\lambda & r \\ p & q & r & d-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^4 - \ldots - (A + B + C + D)\lambda + \Delta = 0. \qquad (76.3)$$

In this case $\bar{F}_\lambda = 0$ also represents a cone for the *same* value of λ, which is now a root of the equation

$$\lambda^4 - \ldots - (\bar{A} + \bar{B} + \bar{C} + \bar{D})\lambda + \bar{\Delta} = 0.$$

Hence $A + B + C + D = \bar{A} + \bar{B} + \bar{C} + \bar{D}$ and $\Delta = \bar{\Delta}$.

But $D = \bar{D}$ and so $A + B + C = \bar{A} + \bar{B} + \bar{C}$.

We have thus established that I, J, D, Δ and $A + B + C$ are invariants under a rotation.

77. General transformation of coordinates

A triple of mutually orthogonal right-handed axes can be moved into coincidence with another triple of mutually orthogonal right-handed axes by a translation to the new origin followed by a suitable rotation of axes. We see from §§74 and 76 that

$$I, \quad J, \quad D, \quad \Delta \qquad (77.1)$$

are invariants for this general coordinate transformation. (Note that $A + B + C$ is not an invariant under translations.)

These invariants are **relative invariants** because they are associated with a quadratic form F and not with a surface $F = 0$. As in the theory of conics†, any function of the invariants, homogeneous of degree zero in $a, b, c, d, f, g, h, p, q, r$ is an **absolute invariant.** An independent set of three absolute invariants could be taken to be IJ/D, I^2/J and ID/Δ. Any geometrical property which is independent of the choice of axes must be expressible in terms of the absolute invariants.

Example 4. Show that the degree of an equation is invariant under translations and rotations of axes.

Example 5. Prove that the roots of the discriminating cubic are relative invariants for translations and rotations of axes.

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† *Analytical Conics*, p. 64.

CHAPTER X

CLASSIFICATION AND REDUCTION OF QUADRIC

78. Equation of quadric

We shall now justify the remark made in section 59 that the general equation of the second degree

$$F \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0 \quad (78.1)$$

always represents a central quadric or paraboloid or one of their degeneracies.

From the theory of conics†, we know that a rotation through the angle $\frac{1}{2}\tan^{-1}\{2h/(a-b)\}$ about the z-axis will remove the product term in xy. Having removed this term, a suitable rotation about the *new* y-axis will remove the product term zx and similarly the term in yz can be removed by a third rotation. After these three rotations the equation of the surface takes the form

$$F' \equiv a'x'^2 + b'y'^2 + c'z'^2 + 2p'x' + 2q'y' + 2r'z' + d' = 0. \quad (78.2)$$

Case I. Suppose $a'b'c' \neq 0$. Then a translation to parallel axes through $(-p'/a',\ -q'/b',\ -r'/c')$ changes the equation of the surface into

$$\bar{F} \equiv \bar{a}\bar{x}^2 + \bar{b}\bar{y}^2 + \bar{c}\bar{z}^2 + \bar{d} = 0. \quad (78.3)$$

In this case (78.1) represents a central quadric.

Case II. It may happen that the successive rotations which caused f', g' and h' to vanish also make one of a', b' or c' zero. There is no loss of generality if we select $c' = 0$ in (78.2). Suppose $r' \neq 0$ and we see that a translation of axes to the new origin $(-p'/a',\ -q'/b',\ -d'/2r' + p'^2/2a'r' + q'^2/2b'r')$ changes the equation of the surface to

$$\bar{F} \equiv \bar{a}\bar{x}^2 + \bar{b}\bar{y}^2 + 2\bar{r}\bar{z} = 0. \quad (78.4)$$

Now (78.1) represents a paraboloid.

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† *Analytical Conics*, p. 66.

It may happen that $r' = 0$ when $c' = 0$. If so, translate to parallel axes through $(-p'/a', -q'/b', 0)$ and the surface is represented by

$$\bar{F} \equiv \bar{a}\bar{x}^2 + \bar{b}\bar{y}^2 + \bar{d} = 0 \tag{78.5}$$

and is therefore an elliptic or hyperbolic cylinder if $\bar{d} \neq 0$ and a pair of planes intersecting in a finite line if $\bar{d} = 0$.

Case III. Two of the coefficients a', b', c' may be zero after the three initial rotations which make $f' = g' = h' = 0$. (Note that a', b', c' cannot all be zero as then (78.2) is no longer quadratic.) Suppose that $b' = c' = 0$. Now translate to parallel axes through $(-p'/a', -d'/2q' + p'^2/2aq', 0)$ and the surface is given by

$$F'' \equiv a''x''^2 + 2q''y'' + 2r''z'' = 0.$$

Rotate about the x''-axis to make the straight line $q''y'' + r''z'' = x'' = 0$ the new $\bar{y}$-axis and our equation reduces to

$$\bar{F} \equiv \bar{a}\bar{x}^2 + 2\bar{q}\bar{y} = 0 \tag{78.6}$$

which represents a parabolic cylinder if $\bar{q} \neq 0$ and a pair of coincident planes if $\bar{q} = 0$.

Case IV. Finally, it may happen that $q' = r' = 0$ as well as $b' = c' = 0$. The equation of the surface is now

$$F' \equiv a'x'^2 + 2p'x' + d' = 0, \tag{78.7}$$

which represents a pair of parallel planes. These planes coincide if $p' = d' = 0$.

We have thus shown that (78.1) represents either a central quadric, paraboloid, cone, elliptic, hyperbolic or parabolic cyclinder or a plane-pair.

Example 1. By making the appropriate translations and rotations, show that the equation $x^2 + y^2 + z^2 + yz + zx + xy + x + y + z - 1 = 0$ represents an ellipsoid, whose equation referred to its principal axis is $16x^2 + 4y^2 + 4z^2 - 11 = 0$.

79. Classification of quadrics

We may classify a quadric by considering the position of its centre which is at the point (P, Q, R, D). The quadric is (see §68):

(i) a central quadric or cone (excluding cylinder) if there exists a *unique finite* centre;

(ii) a paraboloid if there exists a *unique* centre at *infinity*;

(iii) an elliptic or hyperbolic cylinder if there exists a *finite line* of centres;

(iv) a parabolic cylinder if there exists a *line* of centres at *infinity*;

(v) a pair of parallel planes if there exists a *plane* of centres.

The polar plane of the centre is the plane at infinity and so the centre lies on the polar planes of the three points at infinity $(1, 0, 0, 0)$, $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$. That is, the coordinates of the centre satisfy the three equations

$$\left.\begin{aligned} \partial F/\partial x &\equiv ax + hy + gz + pw = 0, \\ \partial F/\partial y &\equiv hx + by + fz + qw = 0, \\ \partial F/\partial z &\equiv gx + fy + cz + rw = 0. \end{aligned}\right\} \qquad (79.1)$$

We have already investigated in section 55 all cases of the intersection of three planes and summarized the results in the table on page 59. For equations (79.1) the corresponding matrices are

$$\boldsymbol{U} \equiv \begin{pmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \end{pmatrix} \text{ and } \boldsymbol{V} \equiv \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \equiv \boldsymbol{D}.$$

Further, we have shown in §72 that the quadric $F = 0$ degenerates to a cone (including cylinder) if $R(\boldsymbol{\Delta}) = 3$, to a plane-pair if $R(\boldsymbol{\Delta}) = 2$ and a coincident pair of planes if $R(\boldsymbol{\Delta}) = 1$.

Since $\boldsymbol{D}$ is a sub-matrix of $\boldsymbol{\Delta}$, we always have $R(\boldsymbol{D}) \leq R(\boldsymbol{\Delta})$. Also, by Example 1, §60, we have that $R(\boldsymbol{\Delta}) \leq 3$ if $R(\boldsymbol{D}) = 1$.

The classification of the quadric now follows:

Case I. If $\partial F/\partial x = 0$, $\partial F/\partial y = 0$, $\partial F/\partial z = 0$ intersect at a unique finite point, then $R(\boldsymbol{D}) = 3$, and the quadric is a central quadric if $R(\boldsymbol{\Delta}) = 4$ but a cone (excluding cylinder) if $R(\boldsymbol{\Delta}) = 3$.

Case II. If the three planes (79.1) intersect at a unique centre at infinity $R(\boldsymbol{D}) = 2$. Accordingly the quadric is a paraboloid if $R(\boldsymbol{\Delta}) = 4$.

Case III. If the three planes (79.1) intersect in a finite line we have $R(\boldsymbol{D}) = 2$ and $R(\boldsymbol{U}) = 2$. Hence $P = Q = R = D = 0$ and so $R(\boldsymbol{\Delta}) = 3$ or 2. If $R(\boldsymbol{\Delta}) = 3$ the quadric is a cone (including cylinder) with a finite line of centres. That is, an elliptic or hyperbolic cylinder. If $R(\boldsymbol{\Delta}) = 2$ the quadric is a plane-pair with a finite line of intersection.

Case IV. If the three planes (79.1) intersect in a line at infinity, the three planes are parallel and we have $R(\boldsymbol{D}) = 1$ and $R(\boldsymbol{U}) = 2$. Thus $R(\boldsymbol{\Delta}) \leq 3$. If $R(\boldsymbol{\Delta}) = 3$ the quadric is a cone (including cylinder) with a line of centres at infinity and so is a parabolic cylinder.

Case V. If the three planes (79.1) coincide, we have $R(\boldsymbol{D}) = 1$ and $R(\boldsymbol{U}) = 1$. Therefore $R(\boldsymbol{\Delta}) = 2$ or 1. The former must correspond to a pair of parallel planes and the latter to a pair of coincident planes.

These results are conveniently summarized in the following table:

		RANK OF MATRIX $\boldsymbol{\Delta}$			
		4	3	2	1
RANK OF MATRIX $\boldsymbol{D}$	3	Central quadric	Cone	✕	✕
	2	Paraboloid	Elliptic or hyperbolic cylinder	Plane-pair intersecting in a finite line	✕
	1	✕	Parabolic cylinder	Parallel plane-pair	Coincident plane-pair

We now proceed to examine the cases individually.

Example 2. Classify the quadrics given by the following equations:

(*a*) $x^2 + 2yz - 4x + 6y + 2z = 0$,
(*b*) $x^2 - z^2 + 2yz - 2xy - x - y + z = 0$,
(c) $2x^2 - y^2 + 3yz - xy - 1 = 0$,
(*d*) $5x^2 + y^2 + z^2 - 4zx + 2xy + 2x + 2y = 0$,
(e) $2x^2 - y^2 - z^2 + 2yz + 8zx - 8xy + 2y + 4z = 0$,
(*f*) $4y^2 - 4yz + 4zx - 4xy - 2x + 2y - 1 = 0$,
(*g*) $27x^2 - 4y^2 + 36z^2 + 72zx + 12xy - 6x + 4y - 1 = 0$,
(*h*) $4x^2 + 9y^2 + 16z^2 + 24yz + 16zx + 12xy + 2x + 4y + 6z + 1 = 0$.

80. Central quadric and cone

The quadric $F = 0$, by rotations and translations, takes the standard form (78.3), namely

$$\bar{a}\bar{x}^2 + \bar{b}\bar{y}^2 + \bar{c}\bar{z}^2 + \bar{d} = 0 \quad (\bar{a}\bar{b}\bar{c} \neq 0).$$

From the theory (§77) of relative invariants, we have

$$\bar{a} + \bar{b} + \bar{c} = a + b + c \equiv I$$
$$\bar{b}\bar{c} + \bar{c}\bar{a} + \bar{a}\bar{b} = \mathscr{A} + \mathscr{B} + \mathscr{C} \equiv J$$
$$\bar{a}\bar{b}\bar{c} = D$$
$$\bar{a}\bar{b}\bar{c}\bar{d} = \Delta.$$

Thus $\bar{a}$, $\bar{b}$, $\bar{c}$ are the roots λ_1, λ_2, λ_3 of the discriminating cubic equation

$$\lambda^3 - I\lambda^2 + J\lambda - D = 0 \tag{80.1}$$

and $\bar{d} = \Delta/D$. We say that the reduced equation of the quadric is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \Delta/D = 0. \tag{80.2}$$

We may drop the bars on x, y, z provided that there is no chance of confusion with the old coordinate system.

The quadric is clearly a surface of revolution if the discriminating cubic has a double root (compare §70).

Example 3. Obtain the reduced equations of the quadrics in Example 2(a) and (c).

81. Paraboloid

We have $D = 0$ and the equation of the quadric can be simplified to (78.4), namely

$$\bar{a}\bar{x}^2 + \bar{b}\bar{y}^2 + 2\bar{r}\bar{z} = 0.$$

The theory of relative invariants now yields

$$\bar{a} + \bar{b} = a + b + c \equiv I$$
$$\bar{a}\bar{b} = \mathscr{A} + \mathscr{B} + \mathscr{C} \equiv J$$
$$-\bar{a}\bar{b}\bar{r}^2 = \Delta.$$

Thus $\bar{a}$, $\bar{b}$ are the two non-zero roots λ_1, λ_2 of the discriminating cubic (80.1), and $\bar{r} = \pm\sqrt{-\Delta/J}$. We say that the reduced equation of the paraboloid is

$$\lambda_1 x^2 + \lambda_2 y^2 = 2\sqrt{-\Delta/J}\, z. \tag{81.1}$$

The reader is asked to verify the identities

$$AD - P^2 = \mathscr{A}\Delta; \quad BD - Q^2 = \mathscr{B}\Delta; \quad CD - R^2 = \mathscr{C}\Delta.$$

For the paraboloid $D = 0$ and so by addition $(\mathscr{A} + \mathscr{B} + \mathscr{C})\Delta < 0$. Hence $J\Delta$ is negative and consequently $\sqrt{-\Delta/J}$ is real.

By considering the paraboloid in its standard form $ax^2 + by^2 + 2rz = 0$, it follows immediately that the normal at the vertex passes through its centre at infinity. The normal at A to (78.1) has the equations $(w_A x - x_A w)/(\partial F_A/\partial x_A) = (w_A y - y_A w)/(\partial F_A/\partial y_A) = (w_A z - z_A w)/(\partial F_A/\partial z_A)$ and so passes through the centre $(P, Q, R, 0)$ if $P/(\partial F_A/\partial x_A) = Q/(\partial F_A/\partial y_A) = R/(\partial F_A/\partial z_A)$. Hence the vertex of a paraboloid is obtained by solving for the finite root of the equations

$$\frac{\partial F}{\partial x}\bigg/ P = \frac{\partial F}{\partial y}\bigg/ Q = \frac{\partial F}{\partial z}\bigg/ R; \quad F = 0. \tag{81.2}$$

Example 4. Obtain the reduced equations of the quadrics in Example 2(*b*) and (*e*).

82. Elliptic and hyperbolic cylinder

The equation of the quadric in this case simplifies to (78.5), namely

$$\bar{a}\bar{x}^2 + \bar{b}\bar{y}^2 + \bar{d} = 0. \tag{82.1}$$

Again the theory of relative invariants shows that $\bar{a}$, $\bar{b}$ are the two non-zero roots of the discriminating cubic (80.1). We cannot compute $\bar{d}$ directly since both D and Δ are zero. It is convenient, however, to consider the intermediate equation (78.2) which has the form

$$a'x'^2 + b'y'^2 + 2p'x' + 2q'y' + d' = 0. \tag{82.2}$$

This equation is derived from (78.1) by rotations without any translations. Therefore, we may use the invariance of $A + B + C$ (compare §76) to yield

$$a'b'd' - b'p'^2 - a'q'^2 = A + B + C.$$

We may write (82.2)

$$a'(x' + p'/a')^2 + b'(y' + q'/b')^2 + d' - p'^2/a' - q'^2/b' = 0.$$

Hence the translation which puts this equation into the form (82.1) requires that

$$\bar{d} = d' - \frac{p'^2}{a'} - \frac{q'^2}{b'} = \frac{A + B + C}{a'b'} = \frac{A + B + C}{\mathscr{A} + \mathscr{B} + \mathscr{C}}.$$

Since $P = Q = R = D = 0$ (§79, Case III), we have by identities of type (60.1) and (60.2) that

$$aA + hH + gG = 0; \quad hA + bH + fG = 0.$$

Elimination of G yields $\mathscr{F}A - \mathscr{G}H = 0$. But $\mathscr{A}\mathscr{F} - \mathscr{H}\mathscr{G} = fD = 0$. Therefore $A/\mathscr{A} = H/\mathscr{H}$. By symmetry we have

$$\frac{A}{\mathscr{A}} = \frac{B}{\mathscr{B}} = \frac{C}{\mathscr{C}} = \frac{F}{\mathscr{F}} = \frac{G}{\mathscr{G}} = \frac{H}{\mathscr{H}} = \frac{A + B + C}{\mathscr{A} + \mathscr{B} + \mathscr{C}},$$

and so the reduced equation of the elliptic or parabolic cylinder is

$$\lambda_1 x^2 + \lambda_2 y^2 + A/\mathscr{A} = 0. \tag{82.3}$$

This case includes for $A/\mathscr{A} = 0$ the case of a plane pair intersecting in a finite straight line.

Example 5. Obtain the reduced equations of the quadrics in Example 2(d) (f) and (g).

83. Parabolic cylinder and parallel plane-pair

It is best to tackle the case $R(\mathbf{D}) = 1$ without invariant theory. The terms of the second degree now form a perfect square. Since $bc = f^2$, $ca = g^2$, $ab = h^2$ it follows that a, b and c have the same sign and we shall assume that the sign is positive. (If not, consider the equation $-F = 0$ of the quadric.) We may, therefore, write the equation of the quadric (78.1) in the form

$$(\sqrt{a}x + \sqrt{b}y + \sqrt{c}z)^2 + 2px + 2qy + 2rz + d = 0.$$

This equation is of the type $\bar{a}\bar{x}^2 + 2\bar{q}\bar{y} = 0$ where $\bar{x} = 0$ and $\bar{y} = 0$ represent planes, but these planes may not be mutually orthogonal. Let us re-write this equation

$$(\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + k)^2$$
$$= 2(k\sqrt{a} - p)x + 2(k\sqrt{b} - q)y + 2(k\sqrt{c} - r)z + k^2 - d, \tag{83.1}$$

and choose k so that the planes

$$u \equiv \sqrt{a}x + \sqrt{b}y + \sqrt{c}z + k = 0,$$
$$v \equiv 2(k\sqrt{a} - p)x + 2(k\sqrt{b} - q)y + 2(k\sqrt{c} - r)z + k^2 - d = 0,$$

are mutually orthogonal. This yields

$$k = (p\sqrt{a} + q\sqrt{b} + r\sqrt{c})/(a + b + c).$$

Now choose $u = 0$ and $v = 0$ as the new coordinate planes $\bar{x} = 0$ and $\bar{y} = 0$. Then

$$\bar{x} = u/\sqrt{a+b+c};$$

$$\bar{y} = v/2\sqrt{(k\sqrt{a}-p)^2 + (k\sqrt{b}-q)^2 + (k\sqrt{c}-r)^2},$$

and the equation of the parabolic cylinder reduces on dropping the bars to

$$x^2 = 4\sigma y$$

where

$$\sigma = \sqrt{(k\sqrt{a}-p)^2 + (k\sqrt{b}-q)^2 + (k\sqrt{c}-r)^2}/2(a+b+c).$$

By Lagrange's identity (6.3) with $l_1 = \sqrt{a}$, $m_1 = \sqrt{b}$, $n_1 = \sqrt{c}$, $l_2 = k\sqrt{a} - p$, $m_2 = k\sqrt{b} - q$, $n_2 = k\sqrt{c} - r$ and noting that $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, we readily obtain

$$\sigma = \frac{1}{2}\left\{\frac{(q\sqrt{c} - r\sqrt{b})^2 + (r\sqrt{a} - p\sqrt{c})^2 + (p\sqrt{b} - q\sqrt{a})^2}{(a+b+c)^3}\right\}^{\frac{1}{2}}. \quad (83.2)$$

From (83.1) we see that the parabolic cylinder reduces to a pair of parallel planes if $k\sqrt{a} - p = k\sqrt{b} - q = k\sqrt{c} - r = 0$, and the reader is asked to verify that the reduced equation is

$$x^2 = (p^2 - ad)/a(a+b+c).$$

Clearly the reduced equation of coincident planes is $x^2 = 0$.

Example 6. Obtain the reduced equation of the quadric in Example 2(h).

Miscellaneous examples

1. Show that the locus of points equidistant from two given skew straight lines is a hyperbolic paraboloid.

2. Prove that the normals to a quadric along a generator, generate a hyperbolic paraboloid whose principal sections are equal parabolae.

3. Find the vertex of the paraboloid $x^2 + (y \cos \alpha + z \sin \alpha)^2 + 2z = 1$ and also obtain its reduced equation.

4. Obtain the equations of the hyperboloids of revolution which pass through the origin and have the two generators $y = z - k = 0$, $x = z + k = 0$. Further, reduce the equation of the hyperboloids to standard form.

5. Two circles, each of radius a, lie in perpendicular planes and touch the common line of these planes at the same point. Show that one paraboloid and one elliptic cylinder pass through the two circles and find their reduced equations.

6. Calculate the semi-vertical angle of the right circular cone $yz + zx + xy = 0$.

7. Show that the equation $\alpha(y - z)^2 + \beta(z - x)^2 + \gamma(x - y)^2 = 1$ represents a cylinder which is hyperbolic when $\beta\gamma + \gamma\alpha + \alpha\beta < 0$ but real and elliptic if $\beta\gamma + \gamma\alpha + \alpha\beta > 0$ and $\alpha + \beta + \gamma > 0$. What happens when $\beta\gamma + \gamma\alpha + \alpha\beta = 0$?

8. If $a + b + c \neq 0$, show that the equation $a(x^2 + 2yz) + b(y^2 + 2zx) + c(z^2 + 2xy) = 1$ represents a hyperboloid. Discuss the case $a + b + c = 0$.

Answers

2. (*a*) Central quadric, (*b*) paraboloid, (*c*) central quadric, (*d*) elliptic or hyperbolic cylinder, (*e*) paraboloid, (*f*) elliptic or hyperbolic cylinder, (*g*) plane pair with finite line of intersection, (*h*) parabolic cylinder. **3.** $x^2 + y^2 - z^2 = 10$, $2x^2 + 3\sqrt{2}y^2 - 3\sqrt{2}z^2 = 2$. **4.** $3x^2 - 3y^2 = z$, $2x^2 - 2y^2 = 2\sqrt{3}z$. **5.** $6x^2 + y^2 = 1$, $6x^2 - 2y^2 = 1$, $68x^2 - 9y^2 = 0$. **6.** $841x^2 = 2\sqrt{174}y$.

Miscellaneous examples. **3.** $(0, -\tan\alpha(1 + \frac{1}{2}\cos^2\alpha), \frac{1}{2}\cos^2\alpha)$, $x^2 + y^2 = 2z\cos\alpha$. **4.** $xy \pm x(z - k) \pm y(z + k) = 0$. $x^2 + y^2 - 2z^2 = 2k^2$. **5.** $x^2 + 2y^2 = 2\sqrt{2}az$, $x^2 + 2y^2 = a^2$. **6.** $\tan^{-1}\sqrt{2}$. **7.** Parallel plane-pair. **8.** Hyperbolic cylinder.

CHAPTER XI

FOCI—CONFOCALS

84. Focus

The locus of a point such that the square of its distance from a fixed point K is proportional to the product of its distances from the two fixed planes $u_1 \equiv \lambda_1 x + \mu_1 y + \nu_1 z + \rho_1 = 0$ and $u_2 \equiv \lambda_2 x + \mu_2 y + \nu_2 z + \rho_2 = 0$ is the quadric

$$\begin{aligned}(x - x_K)^2 + (y - y_K)^2 + (z - z_K)^2 \\ - k(\lambda_1 x + \mu_1 y + \nu_1 z + \rho_1)(\lambda_2 x + \mu_2 y + \nu_2 z + \rho_2) = 0. \quad (84.1)\end{aligned}$$

We call K a **focus** and the line of intersection of the planes $u_1 = 0$ and $u_2 = 0$ is called the **directrix** corresponding to the focus K.

We may now ask: "Has the quadric $F = 0$ any foci?" If so, we can write

$$F \equiv \lambda\{(x - x_K)^2 + (y - y_K)^2 + (z - z_K)^2 - ku_1u_2\}. \quad (84.2)$$

Hence the quadric

$$\Phi \equiv F - \lambda\{(x - x_K)^2 + (y - y_K)^2 + (z - z_K)^2\} \equiv -\lambda k u_1 u_2 = 0$$

is the plane-pair $u_1u_2 = 0$. The necessary and sufficient condition that $\Phi = 0$ be a plane-pair (see §72) is that the 4×4 matrix formed by the coefficients of Φ has rank two. In the next sections we obtain the foci of the central quadrics and paraboloids.

It is of interest to note that the planes $u_1 = 0$ and $u_2 = 0$ intersect the quadric given by (84.1) in circular sections.

Translate axes so that K is the new origin. Let the equation of the quadric become $F' = 0$. By (84.1) we can write

$$F' \equiv \lambda(x^2 + y^2 + z^2 - ku_3u_4)$$

where x, y, z are the coordinates referred to the new axes and $u_3 \equiv \lambda_3 x + \mu_3 y + \nu_3 z + \rho_3$, $u_4 \equiv \lambda_4 x + \mu_4 y + \nu_4 z + \rho_4$.

The tangent cone from the origin to the quadric is given by

$$-k\rho_3\rho_4(x^2 + y^2 + z^2 - ku_3u_4) - \tfrac{1}{4}k^2(\rho_3u_4 + \rho_4u_3)^2 = 0,$$

which reduces to

$$x^2 + y^2 + z^2 - k(\rho_3 u_4 - \rho_4 u_3)^2/4\rho_3\rho_4 = 0.$$

The plane $\rho_3 u_4 - \rho_4 u_3 = 0$ passes through the origin and so a rotation of axes exists for which this equation transforms into

$$X^2 + Y^2 + (1 - \alpha)Z^2 = 0.$$

This equation represents a right circular cone with its axis along the z-axis.

Thus, we have proved that the tangent cones from the foci to a quadric are right circular.

Example 1. Prove that the locus of a point such that its distance from a fixed point equals its distance from a fixed plane cannot be a central quadric.

85. Foci of central quadric

If K is a focus of the central quadric $ax^2 + by^2 + cz^2 + d = 0$, then the quadric

$$\Phi \equiv ax^2 + by^2 + cz^2 + d - \lambda\{(x - x_K)^2 + (y - y_K)^2 + (z - z_K)^2\} = 0$$

is a plane-pair and so

$$R\begin{pmatrix} a-\lambda & 0 & 0 & \lambda x_K \\ 0 & b-\lambda & 0 & \lambda y_K \\ 0 & 0 & c-\lambda & \lambda z_K \\ \lambda x_K & \lambda y_K & \lambda z_K & d - \lambda({x_K}^2 + {y_K}^2 + {z_K}^2) \end{pmatrix} = 2.$$

Hence $\lambda = a$ or b or c.

Consider $\lambda = a$ and equate the right-hand top 3×3 determinant to zero yielding $x_K = 0$. Finally equate the right-hand bottom 3×3 determinant to zero. On reduction the result is

$$\frac{ab{y_K}^2}{d(b-a)} + \frac{ca{z_K}^2}{d(c-a)} = 1.$$

Thus every point on the conic

$$x = 0; \quad \frac{aby^2}{d(b-a)} + \frac{caz^2}{d(c-a)} = 1$$

is a focus of the central quadric.

Similarly $\lambda = b$ and $\lambda = c$ yield respectively that every point on the conics

$$y = 0; \quad \frac{bcz^2}{d(c-b)} + \frac{abx^2}{d(a-b)} = 1,$$

$$z = 0; \quad \frac{cax^2}{d(a-c)} + \frac{bcy^2}{d(b-c)} = 1$$

are also foci. These three conics are called the **focal conics.**

There is no loss in generality, if we put $d = -1$. Further let us write† $a = 1/A$, $b = 1/B$, $c = 1/C$ and the equations of the focal conics take the neat forms

$$x = 0; \quad \frac{z^2}{C-A} - \frac{y^2}{A-B} = 1, \tag{85.1}$$

$$y = 0; \quad \frac{x^2}{A-B} - \frac{z^2}{B-C} = 1, \tag{85.2}$$

$$z = 0; \quad \frac{y^2}{B-C} - \frac{x^2}{C-A} = 1. \tag{85.3}$$

Since $(B - C) + (C - A) + (A - B) = 0$, it follows that two of the three quantities $(B - C)$, $(C - A)$, $(A - B)$ are positive and one negative or two negative and one positive. Thus the focal conics comprise an ellipse, a hyperbola and a virtual ellipse. For example, if $B - C$ and $A - B$ are positive whilst $C - A$ is negative the focal conics are respectively a virtual ellipse, a hyperbola and an ellipse.

Example 2. Show that the directrix corresponding to the focus K lying on the conic (85.3) is the straight line $x = Ax_K/(A - C)$; $y = -By_K/(C - B)$.

Example 3. Show that the focal conics of the quadric of revolution $ax^2 + ay^2 + cz^2 = 1$ are the circle $z = 0$; $x^2 + y^2 = (c - a)/ca$, and the axis of rotation $x = y = 0$.

Example 4. Prove that the focal hyperbola passes through the real foci of the focal ellipse and vice-versa.

Example 5. Prove that the real foci of a cone lie on the two straight lines through the vertex.

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† A, B and C are not to be confused with their previous designation as certain cofactors.

86. Foci of paraboloid

If K is a focus of the paraboloid $ax^2 + by^2 + 2rz = 0$, the quadric

$$\Phi \equiv ax^2 + by^2 + 2rz - \lambda\{(x - x_K)^2 + (y - y_K)^2 + (z - z_K)^2\} = 0$$

is a plane-pair and so

$$R\begin{pmatrix} a-\lambda & 0 & 0 & \lambda x_K \\ 0 & b-\lambda & 0 & \lambda y_K \\ 0 & 0 & -\lambda & \lambda z_K + r \\ \lambda x_K & \lambda y_K & \lambda z_K + r & -\lambda(x_K{}^2 + y_K{}^2 + z_K{}^2) \end{pmatrix} = 2.$$

Hence $\lambda = a$ or b. Note that $\lambda = 0$ leaves the matrix with rank *four* and must therefore be excluded.

Consider $\lambda = a$. The right-hand top 3×3 determinant yields $x_K = 0$, whilst the vanishing of the right-hand bottom 3×3 determinant yields on reduction

$$y_K{}^2 = \frac{r(b-a)}{a^2 b}(2az_K + r).$$

Thus every point on the parabola

$$x = 0; \quad y^2 = \frac{r(b-a)}{a^2 b}(2az + r)$$

is a focus of the paraboloid.

Similarly $\lambda = b$ leads to the result that every point on the parabola

$$y = 0; \quad x^2 = \frac{r(a-b)}{ab^2}(2az + r)$$

is also a focus of the paraboloid. The two conics are called the **focal parabolae** of the paraboloid.

There is no loss in generality if we select $r = -1$. Again write $a = 1/A$ and $b = 1/B$ and the equations of the focal parabolae become

$$x = 0; \quad y^2 = (B - A)(2z - A), \tag{86.1}$$

$$y = 0; \quad x^2 = (A - B)(2z - B). \tag{86.2}$$

Example 6. Show that the focal parabolae of the paraboloid of revolution degenerate to the axis of rotation.

87. Confocal central quadrics

The central quadrics are said to be **confocal** when they have the same three focal conics. From equations (85.1), (85.2) and (85.3) it is clear that the central quadrics

$$\frac{x^2}{A-k}+\frac{y^2}{B-k}+\frac{z^2}{C-k}=1 \tag{87.1}$$

form a confocal system of quadrics. Let us assume that $A > B > C$. Then the quadric (87.1) is virtual for $k > A$, a hyperboloid of two sheets for $A > k > B$, a hyperboloid of one sheet for $B > k > C$ and a real ellipsoid for $k < C$. The limiting cases $k = A$, B and C yield the focal conics.

We now ask: "How many confocals of (87.1) pass through a given point P?" The answer is afforded by the solutions of the equation obtained by substituting the coordinates of P in equation (87.1). This equation can then be written

$$\begin{aligned}\psi(k) \equiv (k-A)(k-B)(k-C) &+ x_P{}^2(k-B)(k-C)\\ &+ y_P{}^2(k-C)(k-A) + z_P{}^2(k-A)(k-B) = 0.\end{aligned} \tag{87.2}$$

We have

k	$-\infty$	C	B	A
$\psi(k)$	$-ve$	$+ve$	$-ve$	$+ve$

Hence the cubic equation $\psi(k) = 0$ has three roots k_1, k_2, k_3 such that

$$k_1 < C < k_2 < B < k_3 < A.$$

That is, in general, *three* confocals pass through a given point, one an ellipsoid, one a hyperboloid of one sheet and the other a hyperboloid of two sheets.

Consider the two confocals (87.1) through P corresponding to k_1 and k_2. Subtract the equations obtained by substituting the coordinates of P in (87.1) corresponding to $k = k_1$ and $k = k_2$. Division by $(k_1 - k_2) \neq 0$ then yields

$$\frac{x_P{}^2}{(A-k_1)(A-k_2)}+\frac{y_P{}^2}{(B-k_1)(B-k_2)}+\frac{z_P{}^2}{(C-k_1)(C-k_2)}=0.$$

We deduce that the tangent planes at P to the two confocals are orthogonal. Consequently the three confocals through a point have a triple of mutually orthogonal tangent planes at the point. We say that the confocals form a **triply orthogonal** system of surfaces.

The plane $\lambda x + \mu y + \nu z + \rho = 0$ is a tangent plane to the quadric (87.1) if

$$\lambda^2(A - k) + \mu^2(B - k) + \nu^2(C - k) - \rho^2 = 0,$$

and so only *one* confocal of a system touches a plane.

Return to equation (87.2) which can be written

$$\begin{aligned}(k - k_1)(k - k_2)(k - k_3) \equiv (k - A)(k - B)(k - C)& \\ + x_P{}^2(k - B)(k - C) + y_P{}^2(k - C)(k - A)& \\ + z_P{}^2(k - A)(k - B) = 0.&\end{aligned}$$

Substitute $k = A$, B and C in turn to obtain

$$x_P{}^2 = \frac{(A - k_1)(A - k_2)(A - k_3)}{(A - B)(A - C)};\ y_P{}^2 = \frac{(B - k_1)(B - k_2)(B - k_3)}{(B - C)(B - A)};$$

$$z_P{}^2 = \frac{(C - k_1)(C - k_2)(C - k_3)}{(C - A)(C - B)}. \tag{87.3}$$

Corresponding to a given point P, equation (87.3) yields a unique set of values of k_1, k_2, k_3. Conversely, given a set of values of k_1, k_2, k_3 equations (87.3) determine the point P except for its octant in space. We say that (k_1, k_2, k_3) are the **elliptic coordinates** of the point P.

Example 7. Prove that two confocals of (87.1) touch a given straight line and that the tangent planes at the points of contact are orthogonal.

Example 8. Show that the poles of a given plane with respect to a confocal system of central quadrics lie on the perpendicular to the given plane at the point of contact with a confocal of the system.

Example 9. What surfaces are represented by the following equations in elliptic coordinates: (*a*) $k_1 + k_2 + k_3 = \text{constant}$, (*b*) $k_2k_3 + k_3k_1 + k_1k_2 = \text{constant}$, (*c*) $k_1k_2k_3 = \text{constant}$?

88. Confocal paraboloids

Two paraboloids are said to be **confocal** if they have the same two focal parabolae. By analogy with (87.1) let us investigate the system of paraboloids $x^2/(A - k) + y^2/(B - k) = 2z$. The focal parabolae by (86.1) and (86.2) are $x = 0$; $y^2 = (B - A)(2z - A + k)$ and $y = 0$; $x^2 = (A - B)(2z - B + k)$. It follows that the paraboloids

$$\frac{x^2}{A - k} + \frac{y^2}{B - k} = 2z - k \tag{88.1}$$

form a confocal system of paraboloids with the two focal parabolae (86.1) and (86.2). Let us assume that $A > B$. Then (88.1) is an elliptic paraboloid, vertex upwards, for $k > A$, a hyperbolic paraboloid for $A > k > B$ and an elliptic paraboloid, vertex downwards, for $k < B$. The limiting cases $k = A$ and B yield the two focal parabolae.

If the confocal (88.1) passes through P, the cubic equation satisfied by k is

$$\begin{aligned}\psi(k) \equiv k(k-A)(k-B) - x_P{}^2(k-B) - y_P{}^2(k-A)&\\ - 2z_P(k-A)(k-B) = 0. \quad &(88.2)\end{aligned}$$

We have

k	$-\infty$	B	A	∞
$\psi(k)$	$-ve$	$+ve$	$-ve$	$+ve$

Hence $\psi(k) = 0$ has three roots k_1, k_2, k_3 such that

$$k_1 < B < k_2 < A < k_3.$$

That is, in general, *three* confocals pass through a point, two being elliptic paraboloids and the third a hyperbolic paraboloid.

The reader is asked to verify that confocal paraboloids form a triply orthogonal system of surfaces and that only *one* confocal of the system (88.1) touches a given plane.

Return to equation (88.2) which can be written

$$\begin{aligned}&(k-k_1)(k-k_2)(k-k_3)\\ &\qquad \equiv k(k-A)(k-B) - x_P{}^2(k-B) - y_P{}^2(k-A)\\ &\qquad - 2z_P(k-A)(k-B) = 0.\end{aligned}$$

Substitute $k = A$ and B in turn and further equate the coefficient of k^2 and the results are

$$x_P{}^2 = \frac{(A-k_1)(A-k_2)(A-k_3)}{B-A};$$

$$y_P{}^2 = \frac{(B-k_1)(B-k_2)(B-k_3)}{A-B};$$

$$z_P = \frac{k_1+k_2+k_3-A-B}{2}.$$

We say that (k_1, k_2, k_3) are the **parabolic coordinates** of the point P.

Example 10. A pencil of parallel planes touches a system of confocal paraboloids. Show that the locus of the points of contact is a straight line which intersects both focal parabolae.

Miscellaneous examples

1. Obtain the real focal conics of the quadric $x^2 + 8yz = 1$.

2. Prove that the umbilics of a confocal system lie on the focal conics.

3. The points L, L' lying on the ellipsoids $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ and $x^2/a'^2 + y^2/b'^2 + z^2/c'^2 = 1$ are said to be **corresponding points** if $x_L/a = x_{L'}/a'$, $y_L/b = y_{L'}/b'$ and $z_L/c = z_{L'}/c'$. For two pairs of corresponding points L, L' and M, M' show that $LM' = L'M$ when the ellipsoids are confocal.

4. Show that the normals from a fixed point to a system of confocal central quadrics generate a quadric cone. Is this result true for a confocal system of paraboloids?

5. Prove that the normals at P to the three quadrics through P confocal with a given central quadric F form with the polar plane of P with respect to F a self-polar tetrahedron.

6. Show that the principal axes of the tangent cone from a point to a quadric are the normals to the three confocals which pass through the vertex of the cone.

7. If a straight line is orthogonal to its polar line, with respect to a central quadric, then it is also orthogonal to its polar line with respect to all the confocal quadrics.

8. If the two quadrics $2fyz + 2gzx + 2hxy = 1$ and $2f'yz + 2g'zx + 2h'xy = 1$ can be placed so that they are confocal, show that $fgh/(f^2 + g^2 + h^2) + f'g'h'/(f'^2 + g'^2 + h'^2) = 0$ and $f^2g^2h^2/(f^2 + g^2 + h^2)^3 + f'^2g'^2h'^2/(f'^2 + g'^2 + h'^2)^3 = 1/27$.

Answers

9. (*a*) Sphere $x^2 + y^2 + z^2 = A + B + C - \varepsilon$, (*b*) Quadric $(B + C)x^2 + (C + A)y^2 + (A + B)z^2 = BC + CA + AB - \varepsilon$, (c) Quadric $BCx^2 + CAy^2 + ABz^2 = ABC - \varepsilon$, where ε is the constant in question.

Miscellaneous examples. 1. $y + z = 4x^2 - 12y^2 - 3 = 0$ and $y - z = 4x^2 + 20y^2 - 5 = 0$.

CHAPTER XII

LINEAR SYSTEMS OF QUADRICS

89. Pencil of quadrics

Consider the two quadrics

$$F \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0,$$

$$F' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy + 2p'x + 2q'y + 2r'z + d' = 0,$$

and set up the equation

$$F + kF' = 0. \qquad (89.1)$$

This equation is of the second degree in x and y and thus represents a quadric which passes through the curve of intersection $F = F' = 0$ of the quadrics F and F'. An arbitrary plane cuts the quadrics F and F' in conics which have *four* common points. Consequently $F + kF' = 0$ represents a system of quadrics, called a **pencil** of quadrics, passing through a space curve of the fourth order, called the **base curve.**

Through any point, not on the base curve, there passes one and only one quadric of the pencil because (89.1) is linear in k.

From Section 64, the line AB touches the quadric (89.1) if

$$(F_A + kF'_A)(F_B + kF'_B) - (T_{AB} + kT'_{AB})^2 = 0. \qquad (89.2)$$

This equation is quadratic in k and so in general two quadrics of a pencil touch a given straight line. If A lies on the base curve, the tangency condition (89.2) reduces to $(T_{AB} + kT'_{AB})^2 = 0$. In this case only one quadric of the pencil touches the straight line and the point of contact is A since all quadrics of the pencil pass through A. If A and B both lie on the base curve, then one quadric of the pencil touches AB at the two points A and B. That is, AB is a generator. Finally AB may be a tangent to the base curve and so touches all quadrics of the pencil. Accordingly one quadric of the pencil has AB as a generator.

The plane $\lambda x + \mu y + \nu z + \rho = 0$ touches the quadric (89.1) if the equation corresponding to (63.1) is satisfied. This equation is cubic in k and so three quadrics of a pencil touch a given plane. This number may be less than three if the plane bears some special relationship to the base curve.

Example 1. Show that the polar planes of a fixed point with respect to the quadrics of a pencil form a pencil of planes.

Example 2. Show that the locus of the polar lines of a fixed line with respect to the quadrics of a pencil is a quadric.

90. Self-polar tetrahedron of a pencil of quadrics

The quadric $F + kF' = 0$ is a cone (including degeneracies) if

$$\begin{vmatrix} a + ka' & h + kh' & g + kg' & p + kp' \\ h + kh' & b + kb' & f + kf' & q + kq' \\ g + kg' & f + kf' & c + kc' & r + kr' \\ p + kp' & q + kq' & r + kr' & d + kd' \end{vmatrix} = 0. \tag{90.1}$$

This equation is of the fourth degree in k and so in general a pencil of quadrics contains four cones. If this equation reduces to an identity all quadrics of the pencil are cones.

The vertex A of the cone corresponding to the root k (see §72) satisfies the equations

$$\frac{\partial F_A}{\partial x_A} + k\frac{\partial F'_A}{\partial x_A} = \frac{\partial F_A}{\partial y_A} + k\frac{\partial F'_A}{\partial y_A} = \frac{\partial F_A}{\partial z_A} + k\frac{\partial F'_A}{\partial z_A} = \frac{\partial F_A}{\partial w_A} + k\frac{\partial F'_A}{\partial w_A} = 0. \tag{90.2}$$

The equation of the polar plane of A with respect to the quadric $F + \alpha F' = 0$ of the pencil is

$$\left(\frac{\partial F_A}{\partial x_A} + \alpha\frac{\partial F'_A}{\partial x_A}\right)x + \left(\frac{\partial F_A}{\partial y_A} + \alpha\frac{\partial F'_A}{\partial y_A}\right)y + \left(\frac{\partial F_A}{\partial z_A} + \alpha\frac{\partial F'_A}{\partial z_A}\right)z + \frac{\partial F_A}{\partial w_A} + \alpha\frac{\partial F'_A}{\partial w_A} = 0.$$

Substitution from (90.2) followed by division by $(\alpha - k)$ yields

$$\frac{\partial F'_A}{\partial x_A}x + \frac{\partial F'_A}{\partial y_A}y + \frac{\partial F'_A}{\partial z_A}z + \frac{\partial F'_A}{\partial w_A} = 0.$$

That is, the polar planes of A with respect to all quadrics of the pencil are identical. Accordingly, the vertices of the four cones of a pencil of quadrics form a self-polar tetrahedron with respect to all quadrics of the pencil.

Example 3. Prove that every quadric of the pencil determined by the two quadrics $y^2 - zx + x = 0$ and $x^2 + y^2 + 2zx = 0$ is a cone. Further, show that the only degenerate cones of the pencil are an elliptic cylinder and a plane-pair.

91. Contact of quadrics

(a) Single contact. If the base curve has a double point with distinct tangents, the quadrics of the pencil all touch these two tangent lines and so have a common tangent plane at the double point. Any two quadrics of the pencil are said to have **single contact** with one another.

(b) Double contact. When the quadric F' degenerates to the plane-pair $uv = 0$ whose line of intersection is not a generator of F, the base curve degenerates to the two conics $F = u = 0$ and $F = v = 0$. These two conics intersect at the points A and B where the line of intersection of their planes cuts F. The tangents to the base curve at A and B touch all quadrics of the pencil and so they have two common tangent lines at both A and B. That is, all quadrics of the pencil have two common tangent planes at A and B. Any two quadrics of the pencil are said to have **double contact** with one another.

(c) Stationary contact. A limiting case of double contact occurs when the line of intersection of the planes $u = 0$ and $v = 0$ touches the quadric $F = 0$. Then the two conics possess only one point of intersection but have a common tangent there. Any two quadrics of the pencil are said to have **stationary contact** with one another.

(d) Triple contact. If the base curve consists of a conic and two intersecting lines which cut the conic in distinct points A and B, all quadrics of the pencil have a common tangent plane at A, B and the point of intersection of the lines. Any two quadrics of the pencil are said to have **triple contact** with one another.

(e) Ring contact. When the quadric F' degenerates to the coincident plane-pair $u^2 = 0$, the base curve becomes the conic $F = u = 0$ repeated twice. Since $\partial(u^2)/\partial\theta = 2u\partial u/\partial\theta = 0$ at all points of the plane $u = 0$, it follows that all the quadrics $F + ku^2 = 0$ have common tangent planes at all points of the base curve. Any two quadrics of the pencil are said to have **ring contact** with one another.

(f) Quadruple contact. Now let F and F' degenerate to the plane-pairs $uv = 0$ and $u'v' = 0$, where the straight lines $u = u' = 0$, $u = v' = 0$, $v = u' = 0$ and $v = v' = 0$ are distinct and form a *skew* quadrilateral. All quadrics of the pencil $uv + ku'v' = 0$ contain these four lines as generators and they form the base curve. All quadrics of the pencil have two common generators at the four vertices of the skew quadrilateral and so have a common tangent plane there. That is, all quadrics of the pencil contain four common generators and have common tangent planes at the four points of intersection of the generators. Any two quadrics of the pencil are said to have **quadruple contact.**

Other cases of contact occur corresponding to the following degeneracies of the base curve: (i) the repeated conic of ring contact degenerates to two straight lines; (ii) a straight line and a space cubic. In this case the straight line must be a chord of the cubic and the quadrics of the pencil have double contact at the points of intersection of the chord and the cubic; (iii) two opposite sides of the skew quadrilateral of quadruple contact coincide; (iv) a quartic curve with a cusp; (v) a conic and two straight lines intersecting each other on the conic; (vi) a space cubic and a tangent to the space cubic.

Example 4. Show that the pencil of quadrics $F + kuv = 0$ contains only two cones or an infinite number of cones.

Example 5. Prove that two quadrics which have ring contact with a third quadric have double contact with one another.

Example 6. Investigate the nature of the section of the quadrics

(i) $x^2 + 2z^2 + 4yz + 4xy - 2x + 4z + 1 = 0,$
$x^2 + 2y^2 + 4zx + 6x + 4y + 1 = 0.$

(ii) $3x^2 + y^2 + 3z^2 - 2zx - 2xy + 4x - 2y + 2z - 1 = 0.$
$x^2 + y^2 + 3z^2 - 2yz + 4x + 1 = 0.$

92. Associated points

Any quadric through seven given points can be written in the form

$$F + k'F' + k''F'' = 0 \tag{92.1}$$

since the conditions that the quadric passes through two other points yield two linear equations in k and k' which have in general a unique solution. However, the three equations $F = F' = F'' = 0$ have *eight* solutions and so every quadric through seven given points also passes through an eighth fixed point. Eight such points are said to be **associated points.**

Example 7. Show that the polar plane of a fixed point with respect to all the quadrics through seven fixed points, passes through another fixed point.

Example 8. $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ are two tetrahedra such that A_1, A_2, A_3, A_4 lie respectively in the planes $B_2B_3B_4$, $B_3B_4B_1$, $B_4B_1B_2$, $B_1B_2B_3$, and B_1, B_2, B_3 lie respectively in the planes $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$. Show that B_4 lies in the plane $A_1A_2A_3$.

Miscellaneous examples

1. Show that a pencil of quadrics having double contact with one another exist through the parabola $x^2 - 4ay - a^2 = z = 0$ and the circle $x^2 + z^2 - a^2 = y = 0$.

How many (*a*) cones, (*b*) paraboloids are there in this pencil? Further, prove that the locus of the centres of all quadrics of the pencil is the parabola $z^2 + 2ay = x = 0$.

2. Prove that a pencil of quadrics in ring contact contains only one cone and one paraboloid. Obtain the condition that the cone of the pencil $F + k(\lambda x + \mu y + \nu z + \rho)^2 = 0$ be a cylinder.

3. Show that two quadric cylinders can be drawn through the curve of intersection of the surfaces $x^2 + y^2 + 2z^2 + 2az = 0$ and $z^2 - 2xy = 0$. Obtain their reduced equations.

4. If a quadric has ring contact with a sphere, show that the tangent planes to the sphere cut the quadric in conics which have a focus at the point of contact.

5. Show that the planes $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$ form a self-polar tetrahedron with respect to the quadric $k_1u_1^2 + k_2u_2^2 + k_3u_3^2 + k_4u_4^2 = 0$.

6. Prove that the quadrics through the vertices of a hexahedron whose opposite faces are given by $u_1 = 0$, $v_1 = 0$; $u_2 = 0$, $v_2 = 0$; $u_3 = 0$, $v_3 = 0$ respectively have the equation $k_1u_1v_1 + k_2u_2v_2 + k_3u_3v_3 = 0$.

Deduce that the vertices of a hexahedron form a set of eight associated points.

7. Obtain the equation of the cone whose vertex is at A and which has the section $u = F = 0$.

Hence, show that this cone cuts the quadric $F = 0$ in another conic which lies in the plane $F_Au - 2u_AT_A = 0$.

Answers

Miscellaneous examples. 1. (*a*) two, (*b*) one. **2.** $\Delta(\mathscr{A}\lambda^2 + \mathscr{B}\mu^2 + \mathscr{C}\nu^2 + 2\mathscr{F}\mu\nu + 2\mathscr{G}\nu\lambda + 2\mathscr{H}\lambda\mu) = D(A\lambda^2 + B\mu^2 + C\nu^2 + D\rho^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu + 2P\lambda\rho + 2Q\mu\rho + 2R\nu\rho)$. **3.** $9x^2 + 6y^2 = a^2$ and $2x^2 + y^2 = a^2$. **7.** $u_A{}^2F + F_Au^2 - 2u_AT_Au = 0$.

CHAPTER XIII

PLANE-COORDINATES

93. Plane-coordinates

The plane $u \equiv \lambda x + \mu y + \nu z + \rho w = 0$ determines and is determined by the ratios $\lambda:\mu:\nu:\rho$. Accordingly, we may say that λ, μ, ν and ρ are **homogeneous plane-coordinates** of the plane $u = 0$. We shall refer to it as the plane $(\lambda, \mu, \nu, \rho)$. It will be convenient to denote the particular plane a by $(\lambda_a, \mu_a, \nu_a, \rho_a)$.

The equation $x_A\lambda + y_A\mu + z_A\nu + w_A\rho = 0$ states that the point A lies on the plane $(\lambda, \mu, \nu, \rho)$. Consequently this linear equation in λ, μ, ν and ρ defines the *bundle* of planes with vertex at A. From another point of view we can say that this linear equation defines the point A. That is, in plane-coordinates a point is to be regarded as the vertex of the bundle formed by all planes through it.

Two linear equations $x_A\lambda + y_A\mu + z_A\nu + w_A\rho = 0$ and $x_B\lambda + y_B\mu + z_B\nu + w_B\rho = 0$ represent respectively the bundle of planes through A and B. The combined equations then select the *pencil* of planes having AB as axis. That is, two linear equations define a straight line regarded as the axis of the pencil of planes through it.

Three linear equations $x_A\lambda + y_A\mu + z_A\nu + w_A\rho = 0$, $x_B\lambda + y_B\mu + z_B\nu + w_B\rho = 0$ and $x_C\lambda + y_C\mu + z_C\nu + w_C\rho = 0$ will represent the plane ABC provided that A, B and C are not collinear. In the latter case, they merely represent the straight line ABC.

Example 1. Show that the equation of the point where the line $u_1 = u_2 = 0$ cuts the plane $u_3 = 0$ is

$$\begin{vmatrix} \lambda & \mu & \nu & \rho \\ \lambda_1 & \mu_1 & \nu_1 & \rho_1 \\ \lambda_2 & \mu_2 & \nu_2 & \rho_2 \\ \lambda_3 & \mu_3 & \nu_3 & \rho_3 \end{vmatrix} = 0.$$

94. Principle of duality

We have considered the equation $u \equiv \lambda x + \mu y + \nu z + \rho w = 0$

from two points of view. In point-coordinates we take λ, μ, ν, ρ as constants and $u = 0$ represents all the points (x, y, z, w) which lie on the given plane $(\lambda, \mu, \nu, \rho)$. On the other hand, in plane-coordinates we take x, y, z, w as constants and $u = 0$ represents all the planes $(\lambda, \mu, \nu, \rho)$ which pass through the fixed point (x, y, z, w). Thus there is a duality between a plane considered as its totality of points and a point considered as the bundle of planes through the point as vertex.

An algebraical result then has two interpretations according as we choose x, y, z, w as point-coordinates or plane-coordinates. Two such results are said to be **dual** to one another, and the principle which allows this is called the **principle of duality.**

A straight line can be considered as either the totality of its range of points or the axis of a pencil of planes and so a straight line is self-dual.

95. Quadric envelope

In §63 we proved that the plane $\lambda x + \mu y + \nu z + \rho w = 0$ is a tangent plane to the quadric

$$F \equiv ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2pxw + 2qyw + 2rzw = 0 \qquad (95.1)$$

if

$$\Sigma \equiv A\lambda^2 + B\mu^2 + C\nu^2 + D\rho^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu + 2P\lambda\rho + 2Q\mu\rho + 2R\nu\rho = 0. \quad (95.2)$$

That is, a quadratic equation in λ, μ, ν and ρ determines all the tangent planes of a quadric. We refer to the totality of tangent planes of a quadric as a **quadric envelope** and it is dual to the quadric regarded as the totality of all its points. We also say that (95.2) is the **tangential equation** of the quadric (95.1). Another mode of expression is that the envelope of the family of planes $\lambda x + \mu y + \nu z + \rho w = 0$ subject to the condition $\Sigma = 0$ is the quadric $F = 0$ and that $\Sigma = 0$ is its tangential equation.

The point equation (95.1) can be obtained from the tangential equation (95.2) by means of the identities

$$\left.\begin{aligned}
&\begin{vmatrix} B & F & Q \\ F & C & R \\ Q & R & D \end{vmatrix} \equiv a\Delta^2; &&\begin{vmatrix} A & G & P \\ G & C & R \\ P & R & D \end{vmatrix} \equiv b\Delta^2; \\
&\begin{vmatrix} A & H & P \\ H & B & Q \\ P & Q & D \end{vmatrix} \equiv c\Delta^2; &&\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \equiv d\Delta^2; \\
&\begin{vmatrix} A & H & P \\ G & F & R \\ P & Q & D \end{vmatrix} \equiv -f\Delta^2; &&\begin{vmatrix} H & B & Q \\ G & F & R \\ P & Q & D \end{vmatrix} \equiv g\Delta^2; \\
&\begin{vmatrix} H & F & Q \\ G & C & R \\ P & R & D \end{vmatrix} \equiv -h\Delta^2; &&\begin{vmatrix} H & B & F \\ G & F & C \\ P & Q & R \end{vmatrix} \equiv -p\Delta^2; \\
&\begin{vmatrix} A & H & G \\ G & F & C \\ P & Q & R \end{vmatrix} \equiv q\Delta^2; &&\begin{vmatrix} A & H & G \\ H & B & F \\ P & Q & R \end{vmatrix} \equiv -r\Delta^2.
\end{aligned}\right\} \quad (95.3)$$

Example 2. Prove (i) $\begin{vmatrix} A & H & G & P \\ H & B & F & Q \\ G & F & C & R \\ P & Q & R & D \end{vmatrix} \equiv \Delta^3$

(ii) the identities (95.3)

Example 3. Obtain the point-equations of the following quadric envelopes:

(i) $$\alpha\mu\nu + \beta\nu\lambda + \gamma\lambda\mu - \rho^2 = 0,$$

(ii) $$\lambda^2 - 2\mu\nu + 2\lambda\rho = 0,$$

(iii) $$\lambda^2 + 2\mu\nu + 2\nu\lambda + \rho^2 = 0.$$

96. Conic

An exceptional case of the quadric envelope arises when the matrix

$$\nabla \equiv \begin{pmatrix} A & H & G & P \\ H & B & F & Q \\ G & F & C & R \\ P & Q & R & D \end{pmatrix}$$

is of rank less than four. From Example 1 of the previous section,

$\nabla = \Delta^3$ and so in this case $\Delta = 0$ and $a, b, c, d, f, g, h, p, q, r$ are not determined by equations (95.3).

When $R(\Delta) = 3$, the quadric $F = 0$ degenerates to a cone which has a *singly* infinite number of tangent planes all passing through the fixed vertex. Dually, we can then state that $R(\boldsymbol{\nabla}) = 3$ must correspond to a *quadric envelope* with a singly infinite number of points all lying in a fixed plane. Since the degree of the equation is two, the quadric envelope degenerates to a conic.

The plane of the conic bears the same relation to the quadric envelope as the vertex of the cone does to the cone. Accordingly, the plane of the conic is given by any three of the consistent equations

$$\frac{\partial\Sigma}{\partial\lambda} = \frac{\partial\Sigma}{\partial\mu} = \frac{\partial\Sigma}{\partial\nu} = \frac{\partial\Sigma}{\partial\rho} = 0. \tag{96.1}$$

To obtain the point equations of the conic, we proceed as follows. Let a solution of (96.1) be $\lambda_1, \mu_1, \nu_1, \rho_1$. Then the equation of the plane of the conic is

$$\lambda_1 x + \mu_1 y + \nu_1 z + \rho_1 w = 0. \tag{96.2}$$

Not all three of the points at infinity on the axes can lie in this plane. Suppose, in particular, that the point at infinity (0, 0, 1, 0) does not lie in this plane. That is, $\nu_1 \neq 0$. The equation of this point is $\nu = 0$ and so the *cylinder* enveloped by all the tangent planes of the conic which are parallel to the z-axis satisfies the tangential equations $\nu = \Sigma = 0$. That is, $\nu = 0$, and

$$A\lambda^2 + B\mu^2 + D\rho^2 + 2H\lambda\mu + 2P\lambda\rho + 2Q\mu\rho = 0. \tag{96.3}$$

The problem has now been reduced to finding the point-equation of the conic with line-equation (96.3) when λ, μ, ρ are interpreted as line-cordinates in two dimensions. The resulting equation is (see *Analytical Conics*, p. 106).

$$\begin{aligned}(BD - Q^2)x^2 + 2(PQ - HD)xy + (DA - P^2)y^2 \\ + 2(HQ - BP)x + 2(HP - AQ)y + AB - H^2 = 0.\end{aligned} \tag{96.4}$$

The point-equations of the conic are (96.2) and (96.4).

This procedure breaks down when $R\begin{pmatrix}A & H & P\\ H & B & Q\\ P & Q & D\end{pmatrix} = 2$. In this case it follows from (95.3) that $R(\mathbf{\Delta}) = 2$ and so the expression factorizes into two linear expressions. In this case the quadric envelope degenerates to two points. This is dual to the fact that $R(\mathbf{\Delta}) = 2$ corresponds to a plane-pair.

If $R(\mathbf{\nabla}) = 1$ the quadric envelope reduces to a pair of coincident points.

The converse problem of obtaining the tangential equation from the two point equations of a conic will be treated in section 98.

Example 4. For what value of k does the tangential equation $\lambda^2 + \mu^2 + k\nu^2 + 4\lambda\mu - 2\lambda\nu + 2\mu\rho - \rho^2 = 0$ represent a conic? Obtain the corresponding point equations.

97. Cone

The tangent planes of a cone all pass through its vertex A and so the tangential equations of the cone $F = 0$ are

$$x_A\lambda + y_A\mu + z_A\nu + w_A\rho = \Sigma = 0. \tag{97.1}$$

Conversely, given the tangential equations (97.1) of a cone, then the cone is in fact the enveloping cone from A to the quadric envelope $\Sigma = 0$. The point-equation of this quadric is $F = 0$ and so the point-equation of the cone is

$$F_A F - T_A{}^2 = 0. \tag{97.2}$$

If $R(\mathbf{\nabla}) = 2$ then Σ factorizes into two linear expressions and so a cone envelope degenerates into two straight lines.

Example 5. Obtain the point equation of the cone $\lambda + 2\mu - \rho = \lambda^3 + 2\mu^2 + \rho^2 + 4\mu\nu = 0$.

98. Joachimsthal's quadratic equation

The two planes a and b establish the pencil of planes $(\tau\lambda_a + \sigma\lambda_b, \tau\mu_a + \sigma\mu_b, \tau\nu_a + \sigma\nu_b, \tau\rho_a + \sigma\rho_b)$. The planes of this pencil which touch the quadric with tangential equation $\Sigma = 0$ correspond to the roots of Joachimsthal's equation

$$\tau^2\Sigma_a + 2\tau\sigma\,\Phi_{ab} + \sigma^2\Sigma_b = 0, \tag{98.1}$$

where

$$\Sigma_a \equiv \Sigma(\lambda_a, \mu_a, \nu_a, \rho_a), \quad \Sigma_b \equiv \Sigma(\lambda_b, \mu_b, \nu_b, \rho_b),$$

$$2\Phi_{ab} \equiv \lambda_a \frac{\partial \Sigma_b}{\partial \lambda_b} + \mu_a \frac{\partial \Sigma_b}{\partial \mu_b} + \nu_a \frac{\partial \Sigma_b}{\partial \nu_b} + \rho_a \frac{\partial \Sigma_b}{\partial \rho_b}$$
$$\equiv \lambda_b \frac{\partial \Sigma_a}{\partial \lambda_a} + \mu_b \frac{\partial \Sigma_a}{\partial \mu_a} + \nu_b \frac{\partial \Sigma_a}{\partial \nu_a} + \rho_b \frac{\partial \Sigma_a}{\partial \rho_a}.$$

Further, introduce Φ_a given by

$$2\Phi_a \equiv \lambda_a \frac{\partial \Sigma}{\partial \lambda} + \mu_a \frac{\partial \Sigma}{\partial \mu} + \nu_a \frac{\partial \Sigma}{\partial \nu} + \rho_a \frac{\partial \Sigma}{\partial \rho}$$
$$= \lambda \frac{\partial \Sigma_a}{\partial \lambda_a} + \mu \frac{\partial \Sigma_a}{\partial \mu_a} + \nu \frac{\partial \Sigma_a}{\partial \nu_a} + \rho \frac{\partial \Sigma_a}{\partial \rho_a}.$$

Then the results of §§61–64 dualize immediately to give the following:

(1) The two planes a and b are conjugate if $\Phi_{ab} = 0$.

(2) The pole of the plane a has the equation $\Phi_a = 0$.

(3) The polar of the point $\lambda x + \mu y + \nu z + \rho w = 0$ has plane-coordinates $(ax + hy + gz + pw, hx + by + fz + qw, gx + fy + cz + rw, px + qy + rz + dw)$.

(4) The point of contact of the tangent plane a has the equation $\Phi_a = 0$.

(5) The plane a cuts the quadric envelope in the section whose tangential equation is

$$\Sigma_a \Sigma - \Phi_a^2 = 0. \tag{98.2}$$

This result follows from the fact that the conic section of a quadric envelope by a plane is the dual of the enveloping cone from a point to a quadric. Further, this answers the problem mentioned at the end of §96.

As an interesting application, let us obtain the tangential equation of the circle at infinity, which is the section of the sphere $x^2 + y^2 + z^2 - w^2 = 0$ by the plane $w = 0$. The tangential equation of the sphere is $\lambda^2 + \mu^2 + \nu^2 - \rho^2 = 0$ whilst the coordinates of the plane at infinity are (0, 0, 0, 1). Hence by (98.2) we have $-(\lambda^2 + \mu^2 + \nu^2 - \rho^2) - (\rho)^2 = 0$. That is, the tangential equation of the circle at infinity is

$$\lambda^2 + \mu^2 + \nu^2 = 0. \tag{98.3}$$

Example 6. Obtain the tangential equation of the conic (i) $x + 2y = x^2 + y^2 + 2zx + 2xy + 3 = 0$, (ii) $x - y - z - 1 = 2x^2 + y^2 - 4xy - 4x + 2y + 3 = 0$.

99. Range of quadrics

Consider the two quadric envelopes

$$\Sigma \equiv A\lambda^2 + B\mu^2 + C\nu^2 + D\rho^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu + 2P\lambda\rho + 2Q\mu\rho + 2R\nu\rho = 0,$$

$$\Sigma' \equiv A'\lambda^2 + B'\mu^2 + C'\nu^2 + D'\rho^2 + 2F'\mu\nu + 2G'\nu\lambda + 2H'\lambda\mu + 2P'\lambda\rho + 2Q'\mu\rho + 2R'\nu\rho = 0$$

and set up the equation

$$\Sigma + k\Sigma' = 0. \tag{99.1}$$

This equation is of the second degree in λ, μ, ν and ρ and so represents a quadric envelope which must touch the common tangent planes of the quadric envelopes Σ and Σ' given by $\Sigma = \Sigma' = 0$. The system of quadric envelopes (99.1) is called a **range of quadrics** and is dual to a pencil of quadrics.

The quadric envelope $\Sigma + k\Sigma' = 0$ will degenerate to a conic if the determinantal equation (90.1) where a, b, c, etc. and a', b', c', etc. are replaced by A, B, C, etc. and A', B', C', etc. respectively is satisfied. Thus, in general, a range of quadrics contains four conics. By duality, the planes of the four conics will form a self-polar tetrahedron for all quadric envelopes of the range.

Example 7. Show that the polar lines of a fixed line with respect to the quadric envelopes of a range generate a quadric.

100. Confocal quadrics

The sections of the two quadrics F and F' by a plane a are in general two conics which intersect in four points. When the conics have double contact, the four points become two point-pairs. A quadric of the pencil $F + kF' = 0$ exists whose section by the plane a is the double line joining the two point-pairs. This quadric has coincident generators and so is a cone. Hence the plane a touches one of the four degenerate cones of the pencil.

That is, a plane which cuts two quadrics in conics which are in double contact must touch one of the degenerate cones of the pencil set up by the two quadrics.

The dual of two conics in double contact is two cones having double contact along two generators. Thus the dual result is that a point from which the two tangent cones to two quadrics are in double contact must lie on one of the degenerate conics of the range formed by the two quadrics.

We proved in section 84 that the tangent cone from a focus to a quadric is right circular and so by §57 it has double contact with the isotropic cone whose vertex is at the focus. Thus, by the previous theorem, the focus must lie on the degenerate conics of the range set up by the quadric and the circle at infinity. Accordingly the focal conics of the quadric envelope $\Sigma = 0$ are the degenerate conics of the range

$$\Sigma + k(\lambda^2 + \mu^2 + \nu^2) = 0. \tag{100.1}$$

Further, this equation clearly represents the system of quadrics confocal with the quadric envelope $\Sigma = 0$.

Example 8. Obtain the point-equations of the real focal conics of the quadric $x^2 + 8yz = 1$.

Example 9. Find the point-equations of the quadrics of the confocal system determined by $4\mu\nu - 2\lambda\rho + \rho^2 = 0$.

Miscellaneous examples

1. Prove that the planes which touch the sphere $x^2 + y^2 + z^2 = 1$ and the hyperbola $x + 1 = y^2 - z^2 - 1 = 0$ also touch two other conics. Obtain their point-equations.

2. Show that $\nu^2 + 4\nu\mu + 4\lambda\rho + 4\mu\rho = 0$ represents a paraboloid envelope and obtain the point-equations of the focal parabolae.

3. Prove that one of the focal conics of the quadric envelope $\alpha(\lambda^2 + 2\mu\nu) + 2\nu\lambda + 2\lambda\mu + \rho^2 = 0$ lies in the plane $y = z$. As α varies, show that this conic determines a system of confocal conics.

4. Obtain the tangential equations of the cylinder generated by the tangents to the quadric envelope $\Sigma = 0$ which are parallel to the x-axis.

5. Find the tangential equations of the diameter of the quadric envelope Σ whose conjugate diametral plane is parallel to the plane $x = 0$.

6. Obtain the tangential equation of a paraboloid envelope which touches the z-axis and touches the quadric envelope Σ at the points of contact of the pair of tangent planes through the x-axis.

Answers

3. (i) $\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2 - 2\beta\gamma yz - 2\gamma\alpha zx - 2\alpha\beta xy + \alpha\beta\gamma = 0$, (ii) $2yz - 2x + 1 = 0$, (iii) $x^2 + y^2 + 2yz - 2xy + 1 = 0$. **4.** -1, $x - y - z - 1 = 2x^2 + y^2 - 4xy - 4x + 2y + 3 = 0$. **5.** $x^2 - 2z^2 + 2yz - 2zx + 2x + 2z + 1 = 0$. **6.** (i) $3\nu^2 - 24\lambda\nu + 12\mu\nu - 4\rho^2 = 0$, (ii) $\lambda^2 + \mu^2 - \nu^2 + 4\lambda\mu - 2\lambda\nu + 2\mu\rho - \rho^2 = 0$. **8.** $y + z = 4x^2 - 12y^2 - 3 = 0$; $y - z = 4x^2 + 20y^2 - 5 = 0$. **9.** $k^3 + k^2(x^2 + y^2 + z^2 - 2x) - k(y^2 + z^2 + 4yz + 4) - 4x^2 + 4yz + 8x = 0$.

Miscellaneous examples. **1.** $y = 0,\ z^2 = 4(x + 1)$; $z = 0,\ 4x^2 + y^2 + 4x = 0$.

2. $2x - 2y + 2z + 1 = 4(x - y)^2 + 4(x + y) - 1 = 0$; $x - y - 2z + 1 = (x - y)^2 + 2(x + y) + 1 = 0$.

4. $\lambda = B\mu^2 + C\nu^2 + D\rho^2 + 2F\mu\nu + 2Q\mu\rho + 2R\nu\rho = 0$.

5. $P\lambda + Q\mu + R\nu + D\rho = A\lambda + H\mu + G\nu + P\rho = 0$.

6. $\Sigma - D(\alpha\lambda + \rho)(\alpha'\lambda + \rho) = 0$, where $H^2 = B(A - D\alpha\alpha')$.

APPENDIX I

MATRICES

1. The array of mn numbers set out in m rows and n columns in the form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called a **matrix.** The first suffix r of an element a_{rs} indicates the row in which it occurs whilst the second suffix s indicates the column. Occasionally we refer to an $m \times n$ matrix if it has m rows and n columns.

If $m = n$, the array is called a **square** matrix of order n. The determinant of a square matrix is defined to be that determinant which has the same array as the matrix.

We may delete a number of rows and columns at will. The resulting matrix is called a **sub-matrix** of the original matrix. To every square sub-matrix there corresponds a determinant.

If we expand a determinant of order $(p + 1)$ in terms of any row or column, it is a linear multiple of $(p + 1)$ determinants of order p. It follows that: if all determinants of order p contained in a matrix are zero, then all determinants of higher order than p are also zero. This result enables us to frame the definition of the rank of a matrix.

If all the determinants of order $(p + 1)$ contained in a matrix are zero, whilst at least one determinant of order p is not zero, then the matrix is said to be of **rank** p.

The greatest possible rank of an $m \times n$ matrix is clearly the smaller of m and n.

It is obvious that the rank of a sub-matrix cannot be greater than the rank of the original matrix.

2. We shall use the notation $R(\mathbf{A})$ to denote the rank of a matrix. Let us discuss the particular 3×3 matrix.

$$\boldsymbol{D}_\lambda \equiv \begin{pmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{pmatrix}.$$

We denote the cofactors of $a-\lambda$, $b-\lambda$, $c-\lambda$, f, g, h in the determinant D_λ by $\mathscr{A}_\lambda, \mathscr{B}_\lambda, \mathscr{C}_\lambda, \mathscr{F}_\lambda, \mathscr{G}_\lambda, \mathscr{H}_\lambda$. Then we can verify that

$$\mathscr{B}_\lambda\mathscr{C}_\lambda - \mathscr{F}_\lambda^2 = (a-\lambda)D_\lambda; \quad \mathscr{C}_\lambda\mathscr{A}_\lambda - \mathscr{G}_\lambda^2 = (b-\lambda)D_\lambda;$$
$$\mathscr{A}_\lambda\mathscr{B}_\lambda - \mathscr{H}_\lambda^2 = (c-\lambda)D_\lambda.$$

If $R(\boldsymbol{D}_\lambda) = 2$, we have

$$\mathscr{B}_\lambda\mathscr{C}_\lambda = \mathscr{F}_\lambda^2; \quad \mathscr{C}_\lambda\mathscr{A}_\lambda = \mathscr{G}_\lambda^2; \quad \mathscr{A}_\lambda\mathscr{B}_\lambda = \mathscr{H}_\lambda^2.$$

Hence

$$(\mathscr{A}_\lambda + \mathscr{B}_\lambda + \mathscr{C}_\lambda)^2 = \mathscr{A}_\lambda^2 + \mathscr{B}_\lambda^2 + \mathscr{C}_\lambda^2 + 2\mathscr{F}_\lambda^2 + 2\mathscr{G}_\lambda^2 + 2\mathscr{H}_\lambda^2,$$

and so

$$\mathscr{A}_\lambda + \mathscr{B}_\lambda + \mathscr{C}_\lambda \neq 0. \tag{A.1}$$

If $R(\boldsymbol{D}_\lambda) = 1$, we have

$$(b-\lambda)(c-\lambda) - f^2 = 0; \quad (c-\lambda)(a-\lambda) - g^2 = 0;$$
$$(a-\lambda)(b-\lambda) - h^2 = 0.$$

Consequently

$$\{(a-\lambda) + (b-\lambda) + (c-\lambda)\}^2$$
$$= (a-\lambda)^2 + (b-\lambda)^2 + (c-\lambda)^2 + 2f^2 + 2g^2 + 2h^2,$$

and so

$$(a-\lambda) + (b-\lambda) + (c-\lambda) \neq 0. \tag{A.2}$$

3. The rank of a matrix is easily obtained by assuming the result that the rank is unaltered by

(1) the interchange of two rows or columns,
(2) the multiplication of a row or column by a non-zero constant,
(3) the addition of any two rows or columns.

APPENDIX II

SOLUTIONS

Chapter I. Miscellaneous examples

1. For the tetrahedron $ABCD$ verify that the point $(\frac{1}{4}(x_A + x_B + x_C + x_D)$, etc.) lies on each of the seven straight lines.

2. By symmetry, the required straight line is $x = y = z$.

3. Select a cube with vertices at $(\pm a, \pm a, \pm a)$.

Chapter II

9. An arbitrary point on the given straight line has the coordinates $(-2 + 3\lambda, -3 - \lambda, 2 + 2\lambda)$. Application of (12.1) yields a quadratic in λ.

10. The pencil of planes through the given straight line is $x - 1 + 2(y + 1) + k\{3(y + 1) + z - 3\} = 0$, etc.

16. Referring to Fig. 10, the equations of PA and QB are given by (17.1). Let $PA = a$, $QB = b$, $PR = \lambda$, $QS = \mu$. Then $A \equiv (-a \sin \alpha, a \cos \alpha, c)$, $B \equiv (b \sin \alpha, b \cos \alpha, -c)$. $R \equiv (-\lambda \sin \alpha, \lambda \cos \alpha, c)$, $S \equiv (\mu \sin \alpha, \mu \cos \alpha, -c)$. Hence the coordinates (x, y, z) of the mid-point of RS are $(\frac{1}{2}(\mu - \lambda) \sin \alpha, \frac{1}{2}(\mu + \lambda) \cos \alpha, 0)$. Accordingly (i) $\lambda - a = \mu - b$ and so $x - \frac{1}{2}(b - a) \sin \alpha = z = 0$; (ii) $\lambda - a + \mu - b = k$ and so $y - \frac{1}{2}(a + b + k) \cos \alpha = z = 0$; (iii) $(\lambda - a)^2 + (\mu - b)^2 = k^2$ and so $(x \operatorname{cosec} \alpha - y \sec \alpha - a)^2 + (x \operatorname{cosec} \alpha + y \sec \alpha - b)^2 - k^2 = z = 0$; (iv) $(\lambda - a)(\mu - b) = k^2$ and so $(x \operatorname{cosec} \alpha - y \sec \alpha + a)(x \operatorname{cosec} \alpha + y \sec \alpha - b) + k^2 = z = 0$.

Miscellaneous examples

1. $R \equiv (x_A + \lambda(x_B - x_A)$, etc.) is an arbitrary point of AB. The straight line joining R to the origin is perpendicular to AB if $(x_B - x_A)[x_A + \lambda(x_B - x_A)] + \text{etc.} = 0$. That is, $\lambda = (\Sigma x_A{}^2 - \Sigma x_B x_C)/AB^2$, etc.

2. The direction-ratios of PR are $\{x_A + \lambda(x_B - x_A) - x_P$, etc.$\}$ where R is an arbitrary point of AB. The orthogonality of PR and AB yields $(x_B - x_A)[x_A + \lambda(x_B - x_A) - x_P] + \text{etc.} = 0$. That is $\lambda\Sigma(x_A - x_B)^2 = \Sigma(x_A - x_B)(x_A - x_P)$. Further, $PR^2 = AP^2 - AR^2$ and so $p^2 = \Sigma(x_A - x_P)^2 - \lambda^2\Sigma(x_B - x_A)^2 = [\Sigma(x_A - x_P)^2\Sigma(x_B - x_A)^2 - \{\Sigma(x_A - x_P)(x_A - x_B)\}^2]/AB^2$. Accordingly the result follows on application of Lagrange's identity. (Compare Eq. (6.3).)

3. The straight line through A perpendicular to the given place is $(x - x_A)/\lambda = (y - y_A)/\mu = (z - z_A)/\nu$. Let the mirror image $\bar{A}$ be at $(x_A + k\lambda, y_A + k\mu, z_A + k\nu)$. The mid-point of $A\bar{A}$ is $(x_A + \frac{1}{2}k\lambda, y_A + \frac{1}{2}k\mu, z_A + \frac{1}{2}k\nu)$ and so $\lambda(x_A + \frac{1}{2}k\lambda) + \mu(y_A + \frac{1}{2}k\mu) + \nu(z_A + \frac{1}{2}k\nu) + \rho = 0$. Consequently $k = -2(\lambda x_A + \mu y_A + \nu z_A + \rho)/(\lambda^2 + \mu^2 + \nu^2)$. Substitution for k in the coordinates of $\bar{A}$ yields the result.

4. The mirror image of a straight line in a plane is the line joining its point of intersection with the plane and the mirror image of any one point of the straight line. (If the straight line is parallel to the plane, the mirror image is the line joining the mirror images of any two points of the given straight line.)

5. The equation $(x - x_A)/l - (y - y_A)/m + k\{(x - x_A)/l - (z - z_A)/n\} = 0$ represents a plane through the given straight line. It is orthogonal to $u = 0$ if $\lambda(1 + k)/l - \mu/m - \nu k/n = 0$. Solve for k and substitute in the above equation of the plane.

6. Let $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \equiv (\lambda_1 x + \mu_1 y + \nu_1 z)(\lambda_2 x + \mu_2 y + \nu_2 z)$ and so $\lambda_1\lambda_2 = a$, $\mu_1\mu_2 = b$, $\nu_1\nu_2 = c$, $\mu_1\nu_2 + \mu_2\nu_1 = 2f$, etc. Further, $(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2)^2/(\lambda_1^2 + \mu_1^2 + \nu_1^2)(\lambda_2^2 + \mu_2^2 + \nu_2^2) = 1/4$. Thus $3\Sigma\lambda_1^2\lambda_2^2 - \Sigma(\lambda_1\mu_2 + \lambda_2\mu_1)^2 + 10\Sigma\lambda_1\lambda_2\mu_1\mu_2 = 0$. Hence, etc.

7. Referring to Fig. 10, let RS, RT be perpendiculars from the variable point R on PA, QB respectively. Now $S \equiv (-\lambda\sin\alpha,\ \lambda\cos\alpha,\ c)$. $T \equiv (\mu\sin\alpha,\ \mu\cos\alpha,\ -c)$. The relationships $RS \perp PA$, $RT \perp QB$ and $RS \perp RT$ yield respectively $-\sin\alpha(x_R + \lambda\sin\alpha) + \cos\alpha(y_R - \lambda\cos\alpha) = 0$, $\sin\alpha(x_R - \mu\sin\alpha) + \cos\alpha(y_R - \mu\cos\alpha) = 0$ and $(x_R + \lambda\sin\alpha)(x_R - \mu\sin\alpha) + (y_R - \lambda\cos\alpha)(y_R - \mu\cos\alpha) + (z_R - c)(z_R + c) = 0$. Thus $\lambda = -x_R\sin\alpha + y_R\cos\alpha$ and $\mu = x_R\sin\alpha + y_R\cos\alpha$. Substitution in the last equation and dropping the suffix R yields the locus $x^2\cos^2\alpha - y^2\sin^2\alpha + (z^2 - c^2)\sec 2\alpha = 0$.

8. The determinant
$$\begin{vmatrix} -2a & a+b-c & c+a-b \\ a+b-c & -2b & b+c-a \\ c+a-b & b+c-a & -2c \end{vmatrix} \equiv 0$$
and so the equation represents a plane-pair.

Let $a(z - x)(x - y) + \text{etc.} \equiv (\lambda_1 x + \mu_1 y + \nu_1 z)(\lambda_2 x + \mu_2 y + \nu_2 z)$. Then $\lambda_1\lambda_2 = -a$, $\mu_1\nu_2 + \mu_2\nu_1 = b + c - a$, etc. Accordingly $(\mu_1\nu_2 - \mu_2\nu_1)^2 = (\nu_1\lambda_2 - \nu_2\lambda_1)^2 = (\lambda_1\mu_2 - \lambda_2\mu_1)^2 = \Sigma a^2 - 2\Sigma bc$. Hence, etc.

9. The volume is $\frac{1}{3}p\,.\,OBC$ where p is the perpendicular distance from A to the plane OBC. The equation of this plane is
$$\begin{vmatrix} x & y & z \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix} = 0$$
and so
$$p = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix} \div \sqrt{\Sigma(y_B z_C - y_C z_B)^2}.$$
Let the direction-cosines of OB and OC be $[l_B, m_B, n_B]$ and $[l_C, m_C, n_C]$. Then $y_B z_C - y_C z_B = OB\,.\,OC(m_B n_C - m_C n_B)$ and so $\sqrt{\Sigma(y_B z_C - y_C z_B)^2} = OB\,.\,OC\sqrt{\Sigma(m_B n_C - m_C n_B)^2} = OB\,.\,OC\sin BOC$ by (6.4) $= 2\,OBC$. Hence, etc.

Chapter III

3. Substitute the coordinates in (18.3) and eliminate p, q, r and d (determinantly).

4. The result merely expresses that $AP \perp BP$ if P is any point on the required sphere.

7. A quadratic in λ is obtained by substitution in $T_{AB} = 0$.

11. $x^2 + y^2 + z^2 - 13 + \lambda(x + y + z - 1) = 0$ is a sphere through the given circle. Its centre is at $(-\lambda/2, -\lambda/2, -\lambda/2)$ and has the radius $\sqrt{3\lambda^2/4 + \lambda + 13}$. Equating this to the perpendicular from the centre to the given plane yields a quadratic in λ. Alternatively, apply (21.1) directly.

12. The point $(1, -1, 1)$ lies on both spheres and the corresponding tangent planes are $x + y + 4z - 4 = 0$ and $7x - 4y + z - 14 = 0$. Hence, etc. Alternatively, the angle θ between the two spheres $F_1 = 0$ and $F_2 = 0$ is $\cos^{-2}\{\alpha_1{}^2 + \alpha_2{}^2 - \beta^2)/2\alpha_1\bar{\alpha}_2\}$, where α_1 and α_2 are the radii and β is the distance between the centres. Hence, etc.

13. Locus is the radical line of the spheres.

14. The coaxal system determined by the spheres is $(2 + 4\lambda)(x^2 + y^2 + z^2) - 2(3 + 7\lambda)x + 2(1 + \lambda)y + 2(1 + 3\lambda)z + 7 + 17\lambda = 0$. The point-spheres correspond to $(3 + 7\lambda)^2 + (1 + \lambda)^2 + (1 + 3\lambda)^2 - (2 + 4\lambda)(7 + 17\lambda) = 0$. That is $\lambda = -1$ or $-1/3$ which yield respectively $(x - 2)^2 + y^2 + (z + 1)^2 = 0$ and $(x - 1)^2 + (y + 1)^2 + z^2 = 0$. Hence, etc.

Miscellaneous examples

1. From a diagram, the square of the distance between the centres equals the *difference* of the squares of the radii.

There is no loss in generality if we choose the line joining the centres of the two spheres as x-axis. Then the spheres which are cut in great circles are $x^2 + y^2 + z^2 + 2p_2x + d_2 = 0$ and $x^2 + y^2 + z^2 + 2p_3x + d_3 = 0$. The conditions are $2p_2{}^2 - 2p_1p_2 + d_1 - d_2 = 0$ and $2p_3{}^2 - 2p_1p_3 + d_1 - d_3 = 0$. The point $(\alpha, 0, 0)$ lies on the cutting sphere if $\alpha^2 + 2p_1\alpha + d_1 = 0$. Elimination of p_1 and d_1 yields the quadratic equation

$$\begin{vmatrix} \alpha^2 & \alpha & 1 \\ 2p_2{}^2 - d_2 & -p_2 & 1 \\ 2p_3{}^2 - d_3 & -p_3 & 1 \end{vmatrix} = 0.$$

Hence, etc.

2. The condition is that $x^2 + y^2 + z^2 + 2p_1x + 2q_1y + 2r_1z + d_1 + k_1(\lambda_1x + \mu_1y + \nu_1z + \rho_1) \equiv x^2 + y^2 + z^2 + 2p_2x + 2q_2y + 2r_2z + d_2 + k_2(\lambda_2x + \mu_2y + \nu_2z + \rho_2)$ for some values of k_1 and k_2. Equating coefficients and eliminating k_1 and k_2 yields the result.

3. Referring to Fig. 10, we have $P \equiv (0, 0. c)$, $Q \equiv (0, 0, -c)$, $A \equiv (-\lambda\sin\alpha, \lambda\cos\alpha, c)$. $B \equiv (\mu\sin\alpha, \mu\cos\alpha, -c)$. The equation of the sphere $ABPQ$ is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ c^2 & 0 & 0 & c & 1 \\ c^2 & 0 & 0 & -c & 1 \\ \lambda^2 + c^2 & -\lambda\sin\alpha & \lambda\cos\alpha & c & 1 \\ \mu^2 + c^2 & \mu\sin\alpha & \mu\cos\alpha & -c & 1 \end{vmatrix} = 0.$$

That is, $x^2 + y^2 + z^2 + x(\lambda - \mu)/2\sin\alpha - y(\lambda + \mu)/2\cos\alpha - c^2 = 0$. The radius is $(\lambda - \mu)^2/4\sin^2\alpha + (\lambda + \mu)^2/4\cos^2\alpha - c^2$ which is constant in virtue of AB being constant.

4. An arbitrary point on the straight line is $(\alpha + lt,\ \beta + mt,\ \gamma + nt)$. Substitution in (18.3) yields a quadratic in t. The vanishing of the discriminant gives the required result.

Choose given point as origin. Then $d = 0$ and centre of spheres is at $(-p, -q, -r)$. Hence, etc.

5. The orthogonality conditions are $2pp_i + 2qq_i + 2rr_i - d - d_i = 0$. Elimination of p, q, r and d between these four equations and (18.3) yields the result.

6. The spheres through the given circle form the coaxal system $x^2 + y^2 + z^2 + 2px + 2qy + 2rz + d + 2k(\lambda x + \mu y + \nu z + \rho) = 0$. The square of the radius is $(p + k\lambda)^2 + (q + k\mu)^2 + (r + k\nu)^2 - d - 2k\rho$. This expression is a minimum for the desired sphere and so $k(\lambda^2 + \mu^2 + \nu^2) = \rho - \lambda p - \mu q - \nu r$. Hence, etc.

7. Select $A \equiv (0, 0, 0)$, $B \equiv (0, y_B, 0)$, $C \equiv (x_C, y_C, 0)$, $D \equiv (x_D, y_D, z_D)$. The four altitudes have the equations $x/z_D(y_C - y_B) = -y/x_Cz_D = z/\{x_C(y_D - y_B) - x_D(y_C - y_B)\}$; $x/y_Cz_D = -(y - y_B)/x_Cz_D = z/(x_Cy_D - x_Dy_C)$; $y = y_C$, $(x - x_C)/z_D = -z/x_D$; $x = x_D$, $y = y_D$. The third and fourth line intersect at $(x_D, y_D, -x_D(x_D - x_C)/z_D)$ provided that $y_C = y_D$. This point lies on the other two lines provided that $x_Cx_D + y_Cy_D = y_By_C$. It follows easily that $AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2$.

The vertices AB, AC, AD, BC, BD, CD are conjugate points with respect to (18.3) if (1) $qy_B + d = 0$; (2) $px_C + qy_C + d = 0$; (3) $px_D + qy_D + rz_D + d = 0$; (4) $y_By_C + px_C + q(y_B + y_C) + d = 0$; (5) $y_By_D + px_D + q(y_B + y_D) + rz_D + d = 0$; and (6) $x_Cx_D + y_Cy_D + p(x_C + x_D) + q(y_C + y_D) + rz_D + d = 0$. (4)–(2) yields $y_C = -q$ and (5)–(3) yields $y_D = -q$ and so $y_C = y_D$. Further (6)–(2)–(3) yields $x_Cx_D + y_Cy_D - d = 0$. Using (1) we have $x_Cx_D + y_Cy_D = -qy_B = y_By_C$. Hence, etc.

Chapter IV

1. Let a generator be $x = 1 + lt$, $y = 2 + mt$, $z = -1 + nt$. Substitution in $x + y + z - 1 = 0$ yields $t = -1/(l + m + n)$ from which $x = (m + n)/(l + m + n)$, $y = (2l + m + 2n)/(l + m + n)$ and $z = -(l + m + 2n)/(l + m + n)$. Next, substitution in $x^2 + y^2 + z^2 + 2z = 0$ gives $3l^2 + m^2 + 5n^2 + 4mn + 6nl + 2lm = 0$. Elimination of l, m, n from this equation and the equations of the generator yields the result.

2. The resulting equation must be homogeneous of second degree in x, y and z and also be satisfied by the two equations of the circle.

3. The results follow from the six equations at the beginning of section 28.

4. If $f = h = 0$, then $m = 0$ and $k(b - a) = l^2$, $k(b - c) = n^2$, etc. Also $k(a + b + c) = (l^2 + n^2)(3\cos^2\alpha - 1)$ and $k(a - 2b + c) = -(l^2 + n^2)$. Hence, etc.

5. If $f = g = h = 0$, then two of l, m, n are zero. Since $a = b$ the choice is $l = m = 0$. Hence, etc.

7. The equations $x - y + 2z = x^2 - 3y^2 + 2z^2 = 0$ and $x - y + 2z = 5yz + 3zx + 2xy = 0$ lead to the same quadratic equation $x^2 + 6xz + 5z^2 = 0$. The equations $x^2 - 3y^2 + 2z^2 = 5yz + 3zx + 2xy = 0$ yield the quartic equation $(x^2 + 6xz + 5z^2)(2x^2 - 2xz + 5z^2) = 0$. Hence, etc. Note carefully that $f = 5/2$, $g = 3/2$, $h = 1$ in the application of (29.2).

9. The results follow from (31.1), (31.2) and the equations at the beginning of §28.

10. Eliminate x_A, y_A, z_A and t between $x_A{}^2 + y_A{}^2 + z_A{}^2 + 2z_A = x_A + y_A + z_A - 1 = 0$, $x = x_A + t$, $y = y_A + 2t$, $z = z_A - t$.

Miscellaneous examples

2. The elimination of z between the linear equation and each of the second degree equations must yield the same quadratic equation in x and y. Hence $(a\nu^2 + c\lambda^2)/g\nu\lambda = 2c\lambda\mu/\nu(\lambda f + \mu g - \nu h) = (b\nu^2 + c\mu^2)/f\mu\nu$. The symmetrical form of the answer is obtained by replacing the middle fraction by the equivalent $\{-\mu^2(a\nu^2 + c\lambda^2) + 2c\lambda^2\mu^2 - \lambda^2(b\nu^2 + c\mu^2)\}/\{-g\mu^2\nu\lambda + \lambda\mu\nu(\lambda f + \mu g - \nu h) - f\lambda^2\mu\nu\}$.

3. The equation is equivalent to $a(x + p/a)^2 + b(y + q/b)^2 + c(z + r/c)^2 = p^2/a + q^2/b + r^2/c - d$. Hence, etc.

4. The "rationalized" equation is $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0$. Hence, etc.

5. Select the fixed line as z-axis and the point of intersection with the fixed plane as origin. The locus is then $x^2 + y^2 = (\lambda x + \mu y + \nu z)^2/(\lambda^2 + \mu^2 + \nu^2)$. Hence, etc.

6. Choose the line of intersection of the planes as z-axis and the perpendicular from the fixed point to this line as x-axis. Let the fixed point be $(a, 0, 0)$ and the fixed planes be $\lambda_i x + \mu_i y = 0$ $(i = 1, 2)$. The straight line $(x - a)/l = y/m = z/n$ intersects these planes at the points $(a\mu_i m/\phi_i, -a\lambda_i m/\phi_i, -a\lambda_i n/\phi_i)$ where $\phi_i = \lambda_i l + \mu_i m$. The coordinates of the mid-point satisfy $2x = am(\mu_1/\phi_1 + \mu_2/\phi_2)$, $2y = -am(\lambda_1/\phi_1 + \lambda_2/\phi_2)$. Hence $mx - ly = am$ and $[2\lambda_1\lambda_2 l + (\lambda_1\mu_2 + \lambda_2\mu_1)m]x = [(\lambda_1\mu_2 + \lambda_2\mu_1)l + 2\mu_1\mu_2 m]y$. Elimination of l/m yields the hyperbolic cylinder $2(\lambda_1 x + \mu_1 y)(\lambda_2 x + \mu_2 y) - 2\lambda_1\lambda_2 ax - (\lambda_1\mu_2 + \lambda_2\mu_1)ay = 0$.

7. The equation of the cone with vertex at A and given elliptical section is $(z_A x - x_A z)^2/a^2 + (z_A y - y_A z)^2/b^2 - (z_A - z)^2 = 0$ by (25.2). Select a new origin at (x_A, y_A, z_A) and choose axes parallel to the original axes. The equation of the cone becomes $z_A{}^2x^2/a^2 + z_A{}^2y^2/b^2 + (x_A{}^2/a^2 + y_A{}^2/b^2 - 1)z^2 - 2y_Az_Ayz/b^2 - 2z_Ax_Azx/a^2 = 0$. The condition that the cone be circular is either $y_Az_A = 0$, $z_A{}^2x_A{}^2/a^4 = (1/a^2 - 1/b^2)z_A{}^2(x_A{}^2/a^2 + y_A{}^2/b^2 - z_A{}^2/b^2 - 1)$ or $z_Ax_A = 0$, $y_A{}^2z_A{}^2/b^4 = (1/b^2 - 1/a^2)z_A{}^2(x_A{}^2/a^2 + y_A{}^2/b^2 - z_A{}^2/a^2 - 1) = 0$. These lead to $x^2/(a^2 - b^2) - z^2/b^2 - 1 = y = 0$ or $y^2/(b^2 - a^2) - z^2/a^2 - 1 = x = 0$ respectively. The former corresponds to real points when $a > b$. (For change of axes, consult equation (73.1).)

Chapter V

1. The result follows from (37.3).

2. Consider the three mutually orthogonal tangent planes $\lambda_i x + \mu_i y + \nu_i z = \sqrt{-d(\lambda_i{}^2/a + \mu_i{}^2/b + \nu_i{}^2/c)}$, $(i = 1, 2, 3)$. The result follows by squaring and adding the three equations.

Alternatively, express the condition that the enveloping cone from any point of the locus is orthogonal.

3. The enveloping cone from any point of the locus is rectangular.

4. The pole of $\alpha x_A x + \beta y_A y + \gamma z_A z + \delta = 0$ with respect to $ax^2 + by^2 + cz^2 + d = 0$ is at $(d\alpha x_A/a\delta, d\beta y_A/b\delta, d\gamma z_A/c\delta)$. But $\alpha x_A{}^2 + \beta y_A{}^2 + \gamma z_A{}^2 + \delta = 0$. Hence, etc.

5. The direction-ratios of the generators (39.1) and (39.2) are respectively

$\{a(\lambda^2 - 1),\ 2b\lambda,\ c(\lambda^2 + 1)\}$ and $\{a(1 - \mu^2),\ -2b\mu,\ c(\mu^2 + 1)\}$. The orthogonality condition is $a^2(\lambda^2 - 1)(\mu^2 - 1) + 4b^2\lambda\mu - c^2(\lambda^2 + 1)(\mu^2 + 1) = 0$. The result follows by considering the parametric coordinates (39.3).

6. Let $(x - a\cos\alpha)/l = (y - b\sin\alpha)/m = z/n$ be a generator. Then $(a\cos\alpha + lt)^2/a^2 + (b\sin\alpha + mt)^2/b^2 - n^2t^2/c^2 \equiv 1$. Thus $l^2/a^2 + m^2/b^2 - n^2/c^2 = 0$ and $l\cos\alpha/a + m\sin\alpha/b = 0$ and so $l = a\sin\alpha$, $m = -b\cos\alpha$, $n = \pm c$. Hence, etc.

7. Referred to parallel axes through K, the cone (40.5) has the equation $a(b - c)x_Kyz + b(c - a)y_Kzx + c(a - b)z_Kxy = 0$. Hence, etc.

8. The normals at A and B intersect if

$$\begin{vmatrix} x_A - x_B & y_A - y_B & z_A - z_B \\ ax_A & by_A & cz_A \\ ax_B & by_B & cz_B \end{vmatrix} = 0.$$

Let the chords be parallel to the direction $\{l, m, n\}$. Then $(x_A - x_B)/l = (y_A - y_B)/m = (z_A - z_B)/n$ and so the mid-point has coordinates $2x = x_A + x_B$, $2y = y_A + y_B$, $2z = z_A + z_B$. Hence $\begin{vmatrix} l & m & n \\ al & bm & cn \\ ax & by & cz \end{vmatrix} = 0$. Further, $\Sigma ax_A{}^2 + d = \Sigma ax_B{}^2 + d = 0$ and so $\Sigma a(x_A - x_B)(x_A + x_B) = 0$. That is $alx + bmy + cnz = 0$. Hence, etc.

9. From (41.2) we have $ax/\lambda = by/\mu = cz/\nu$. Hence, etc.

10. The centre of the section by the tangent plane at P is at $x = -\alpha x_P\delta/a\kappa$, etc., where $\kappa = \Sigma\alpha^2x_P{}^2/a$. Substitution in $\Sigma\alpha x_P{}^2 + \delta = 0$ yields $\Sigma a^2x^2/\alpha + \delta^3/\kappa^2 = 0$. Further, $ax^2 + by^2 + cz^2 = \delta^2/\kappa$. Hence, etc.

11. For the tangent plane at A to the cone, we have $\lambda = x_A/(b + c)$, $\mu = y_A/(c + a)$, $\nu = z_A/(a + b)$, $\rho = 0$. The coefficient of R^2 in (42.3) after multiplication by $(aR^2 + d)(bR^2 + d)(cR^2 + d)$ is $d\Sigma x_A{}^2/(b + c) = 0$ and so the roots in R^2 are equal but opposite in sign. Hence, etc.

12. Equation (42.3) reduces to $(R^2 - 1)^2 = 0$. Note carefully that when $\lambda = 0$, one root of (42.3) is given by $aR^2 + d = 0$.

13. Verify that $\kappa = -7/12$ and that the equal roots in R^2 are $1 + 12\rho^2/7$ corresponding to the planes $\sqrt{2}y \pm \sqrt{5}z + \rho = 0$.

14. $\kappa = 3$ and (42.3) reduces to $27R^4 - 12(a^2 + b^2 + c^2)R^2 + 4(b^2c^2 + c^2a^2 + a^2b^2) = 0$. Hence $R_1{}^2R_2{}^2 = 4(b^2c^2 + c^2a^2 + a^2b^2)/27$, etc.

Miscellaneous examples

1. $u_1 + ku_2 = 0$ is a tangent plane if $\Sigma\lambda_1{}^2/a + 2k\Sigma\lambda_1\lambda_2/a + k^2\Sigma\lambda_2{}^2/a = 0$. The required planes are $(u_1 + k_1u_2)(u_1 + k_2u_2) \equiv u_1{}^2 + (k_1 + k_2)u_1u_2 + k_1k_2u_2{}^2 = 0$ where k_1 and k_2 are the roots of the quadratic. Hence, etc.

2. The equation $a(x_A + lt)^2 + b(y_A + mt)^2 + c(z_A + nt)^2 + d = 0$ has equal roots. Hence, etc.

3. The centre (x, y, z) is at $(-\lambda\rho/a\kappa,\ -\mu\rho/b\kappa,\ -\nu\rho/c\kappa)$ and $\rho^2/(\lambda^2 + \mu^2 + \nu^2) = p^2$. Thus $\rho^4 = \kappa^2(a^2x^2 + b^2y^2 + c^2z^2)p^2$ and $\rho^2/\kappa = ax^2 + by^2 + cz^2$. Elimination of κ yields the result.

4. The coefficient of R^4 in (43.2) is proportional to $bc\lambda^2 + ca\mu^2 + ab\nu^2$. This expression must be zero for a parabolic section. Hence, etc.

5. The normal at the point (39.3) is

$$\frac{x - a(\lambda + \mu)/(1 + \lambda\mu)}{(\lambda + \mu)/a(1 + \lambda\mu)} = \frac{y - b(1 - \lambda\mu)/(1 + \lambda\mu)}{(1 - \lambda\mu)/b(1 + \lambda\mu)} = \frac{z - c(\lambda - \mu)/(1 + \lambda\mu)}{-(\lambda - \mu)/c(1 + \lambda\mu)}.$$

This line intersects the plane $z = 0$ at $x = (a^2 + c^2)(\lambda + \mu)/a(1 + \lambda\mu)$, $y = (b^2 + c^2)(1 - \lambda\mu)/b(1 + \lambda\mu)$. Elimination of λ yields the straight line $2a(b^2 + c^2)\mu x + b(a^2 + c^2)(1 - \mu^2)y - (a^2 + c^2)(b^2 + c^2)(1 + \mu^2) = 0$. The envelope corresponds to equating the discriminant of this equation to zero.

6. Let the plane section be $\lambda x + \mu y + \nu z = 0$. From (42.4) we have $l = k\lambda/(aR^2 + d)$ and so $\lambda^2 = l^2(aR^2 + d)^2/k^2$. Substitution in (42.3) yields $\Sigma l^2(aR^2 + d) = 0$ from which $R^2 = -d\Sigma l^2/\Sigma a l^2$. Further, $\Sigma l(aR^2 + d)x = 0$, i.e. $R^2\Sigma lax + d\Sigma lx = 0$. Elimination of R^2 yields the result.

7. The plane joining the origin to the generator (39.1) has the equation $x/a + z/c - \lambda^2(x/a - z/c) = 2\lambda y/b$. The generators (39.1) and (39.2) are parallel if $\lambda\mu + 1 = 0$. Verify that (39.2) lies in the plane for $\mu = -1/\lambda$.

Chapter VI

1. This result follows immediately from 45(vii).

2. The three planes are $2\lambda_i\nu_i x + 2\mu_i\nu_i y + 2\nu_i^2 z = r(\lambda_i^2/a + \mu_i^2/b)$, $(i = 1, 2, 3)$. Addition yields the result.

3. The direction-ratios of the generators (47.1) and (47.2) are $\{a/\lambda, b/\lambda, 2\}$ and $\{a/\mu, -b/\mu, 2\}$. The orthogonality condition is $(a^2 - b^2)/\lambda\mu + 4 = 0$. Hence, etc.

4. The normals are $(x - 2r)/2ar = y/0 = (z + 2ar)/r$ and $x/0 = (y - 2r)/2br = (z + 2br)/r$. The shortest distance can be calculated by (15.3).

5. The centre of the section by the plane $\lambda x + \mu y + \nu z + \rho = 0$ is at $x = \lambda r/a\nu$, $y = \mu r/b\nu$ and $z = -\rho/\nu - r(\lambda^2/a + \mu^2/b)/\nu^2$. Further, $\rho^2/(\lambda^2 + \mu^2 + \nu^2) = p^2$. Elimination of λ/ν, μ/ν and ρ/ν yields the result.

6. Let the plane be $\lambda x + \mu y + \nu z = 0$ where λ, μ, ν are direction-*cosines*. Then $\nu = \sin\alpha$ and so $\lambda^2 + \mu^2 = \cos^2\alpha$. Equation (50.3) reduces to $(R^2 - r^2\cot^2\alpha)(R^2 - r^2\cot^2\alpha\operatorname{cosec}^2\alpha) = 0$. Hence, etc.

7. The coefficient of R^4 in the quadratic (50.3) is proportional to $ab\nu^2$. But $\nu = 0$. Hence, etc.

Miscellaneous examples

1. The generator (47.1) has the parametric representation $(a\lambda + at, -b\lambda + bt, 2\lambda t)$. The foot of the perpendicular from the origin corresponds to $a(a\lambda + at) + b(-b\lambda + bt) + 2\lambda(2\lambda t) = 0$. That is, $t = (b^2 - a^2)\lambda/(a^2 + b^2 + 4\lambda^2)$. Obtain x, y and z in terms of t and λ and the result follows on elimination of λ and t. The negative signs correspond to the generators (47.2).

2. Equation (50.3) has equal roots $-\omega/b^2$.

3. The enveloping cone from any point on the locus is rectangular.

4. After clearing fractions, the coefficient of R^2 in (50.3) is $\omega[\lambda^2 b + \mu^2 a + \nu^2(a + b)]$. Note that ω can only vanish for a tangent plane.

5. Equation (50.3) is $ab\nu^2R^4 + \ldots + (\lambda^2 + \mu^2 + \nu^2)\omega^2 = 0$. Thus

$R_1^2R_2^2 = (\lambda^2 + \mu^2 + \nu^2)\omega^2/ab\nu^2 = c^4$. Further, $x = \lambda r/a\nu$, $y = \mu r/b\nu$, $z = -\rho/\nu - r(\lambda^2/a + \mu^2/b)/\nu^2$. Elimination of λ, μ, ν, ρ and ω yields the result.

6. The straight line $(\alpha + lt, \beta + mt, \gamma + nt)$ touches the paraboloid if the equation $a(\alpha + lt)^2 + b(\beta + mt)^2 + 2r(\gamma + nt) = 0$ has equal roots in t. Hence, etc.

Chapter VII

2. The points A, B and the points at infinity on both straight lines are coplanar.

Miscellaneous examples

2. (*a*) When $0 < a < b < c$, the real planes of circular section are $\sqrt{b - a}\,x \pm \sqrt{c - b}\,z + \rho = 0$. Corresponding to the tangent planes, the poles $(\sqrt{b - a}\,d/a\rho, 0, \pm \sqrt{c - b}\,d/c\rho)$ lie on the plane itself. Thus $\rho^2 = db(a - c)/ac$. Hence, etc.

3. The real planes of circular section are $\pm \sqrt{b - a}\,y + \sqrt{a}\,z + \rho = 0$ with pole at $(0, \pm \sqrt{b - a}\,r/\sqrt{a}b, \rho/\sqrt{a})$. Hence $(b - a)r/\sqrt{a}b + 2\rho = 0$, etc.

Chapter VIII

2 & 3. Note carefully that $f = g = h = \frac{1}{2}$, $d = -1$ or $f = g = h = 1$ and $d = -2$.

4. Select cone with vertex at the origin.

6. The enveloping cone from A to the quadric is $(y_Az_A + z_Ax_A + x_Ay_A + 1)(yz + zx + xy + 1) - \{\frac{1}{2}(y_A + z_A)x + \frac{1}{2}(z_A + x_A)y + \frac{1}{2}(x_A + y_A)z + 1\}^2 = 0$. The intersection with the plane $z = 0$ is the conic $(y_Az_A + z_Ax_A + x_Ay_A + 1)(xy + 1) - \{\frac{1}{2}(y_A + z_A)x + \frac{1}{2}(z_A + x_A)y + 1\}^2 = 0$. The condition for a parabola is $(y_A + z_A)^2(z_A + x_A)^2 = (z_A^2 - y_Az_A - z_Ax_A - x_Ay_A - 2)^2$. Thus locus is $(z^2 - 1)(yz + zx + xy + 1) = 0$ and so the required locus is $z = \pm 1$.

7. The cylinder is given by $F_AF - T_A^2 = 0$ with $A \equiv (1, -1, 0, 0)$ and $F \equiv x^2 + y^2 + 2zx - w^2 = 0$.

8. (*a*) A quadric with $y = z = 0$ and $z - 1 = x + 1 = 0$ as generators has the equation $\alpha y(z - 1) + \beta y(x + 1) + \gamma z(z - 1) + \delta z(x + 1) = 0$. This equation is an identity in z for $x = 2$, $y = -3$. Hence $\alpha = \delta = 3\beta$, $\gamma = 0$, etc.

(*b*) A quadric with first two lines as generators is $\alpha(x + y + z - 1)(x - y + 2z - 3) + \beta(x + y + z - 1)(x + y - 3z - 4) + \gamma(x + 2y - z - 2)(x - y + 2z - 3) + \delta(x + 2y - z - 2)(x + y - 3z - 4) = 0$. The third generator is $(1 + \lambda, -1 - \lambda, 2\lambda)$. Substitution yields an identity in λ from which $\alpha + 4\beta + 3\gamma + 12\delta = 8\alpha + 2\beta + 15\gamma - 30\delta = 2\alpha - 2\beta - 3\gamma + 3\delta = 0$. Hence $\alpha/6 = \beta/21 = -\gamma/14 = -\delta/4$, etc.

9. The two straight lines intersect. With $u_1 = x + y$, $u_2 = y + z$, $u_3 = x + y + z - 1$ we obtain $u_4 = 4u_1 + u_2 - 2u_3$. The second line may be replaced by $u_3 = 0$ and $4u_1 + u_2 = 4x + 5y + z = 0$. Hence a quadric through the two lines has the equation $\alpha(x + y)(x + y + z - 1) + \beta(x + y)(4x + 5y + z) + \gamma(y + z)(x + y + z - 1) + \delta(y + z)(4x + 5y + z) + \varepsilon(4x + 5y + z) = 0$. Putting $z = 0$ and equating coefficients yields $\alpha + 4\beta = 0$, $(\alpha + 5\beta + \gamma + 5\delta)/5 = -(-\alpha + 4\varepsilon)/16$, $2\alpha + 9\beta + \gamma + 4\delta = 0$, $-\alpha - \gamma + 5\varepsilon = 0$. A solution is $\alpha = -12$, $\beta = 3$, $\gamma = 17$, $\delta = -5$, $\varepsilon = 1$. Hence, etc.

Alternatively: The general quadric through the parabola $5y^2 - 16x = z = 0$ is $5y^2 - 16x + z(\lambda x + \mu y + \nu z + \rho) = 0$. $x = -y$, $z = -y$ is a generator and so the identity in y yields $5 + \lambda - \mu + \nu = 0$, $\rho = 16$. The parametric form of the second generator is $(1 - 4t, -1 + 3t, 1 + t)$. Substitution yields the identity $5(-1 + 3t)^2 - 16(1 - 4t) + (1 + t)\{\lambda(1 - 4t) + \mu(-1 + 3t) + \nu(1 + t) + 16\} = 0$. Hence, etc.

10. The polar line of $(x - x_A)/l = (y - y_A)/m = (z - z_A)/n$ with respect to $ax^2 + by^2 + cz^2 + d = 0$ is $T_A = T_B = 0$ where $B \equiv (l, m, n, 0)$. Consequently, the direction-ratios are $\{bc(ny_A - mz_A), \text{etc.}\}$. The orthogonality condition is $\Sigma bcl(ny_A - mz_A) = 0$. The elimination of l, m and n yields the result.

11. The result follows immediately from (67.1).

12. The polar line of a diameter is the line at infinity in the conjugate diametral plane and so intersects all the conjugate diameters.

13. Let $[l_i, m_i, n_i]$ be direction-*cosines* of the conjugate diameters and $\alpha_1, \alpha_2, \alpha_3$ be the lengths of the corresponding semi-diameters. Then $\alpha_1{}^2\Sigma l_1{}^2/a^2 = 1$, $\alpha_2{}^2\Sigma l_2{}^2/a^2 = 1$ and $\alpha_3{}^2\Sigma l_3{}^2/a^2 = 1$. Equations (68.2) are now $\Sigma l_i l_j/a^2 = 0$ for $i \neq j$, $i, j = 1, 2, 3$. Thus $[\alpha_1 l_1/a, \alpha_1 m_1/b, \alpha_1 n_1/c]$, $[\alpha_2 l_2/a, \alpha_2 m_2/b, \alpha_2 n_2/c]$ and $[\alpha_3 l_3/a, \alpha_3 m_3/b, \alpha_3 m_3/c]$ form the direction-cosines of three mutually orthogonal directions. Accordingly $[\alpha_1 l_1/a, \alpha_2 l_2/a, \alpha_3 l_3/a]$, $[\alpha_1 m_1/b, \alpha_2 m_2/b, \alpha_3 m_3/b]$ and $[\alpha_1 n_1/c, \alpha_2 n_2/c, \alpha_3 n_3/c]$ also form the direction-cosines of three mutually orthogonal directions (compare §75). Thus $\alpha_1{}^2 l_1{}^2 + \alpha_2{}^2 l_2{}^2 + \alpha_3{}^2 l_3{}^2 = a^2$, etc. Hence, etc.

14. From the preceding example, note that $\alpha_1{}^2 l_1 m_1 + \alpha_2{}^2 l_2 m_2 + \alpha_3{}^2 l_3 m_3 = 0$. The volume V (see Example 9, p. 19) is given by $\begin{vmatrix} \alpha_1 l_1 & \alpha_1 m_1 & \alpha_1 n_1 \\ \alpha_2 l_2 & \alpha_2 m_2 & \alpha_2 n_2 \\ \alpha_3 l_3 & \alpha_3 m_3 & \alpha_3 n_3 \end{vmatrix}$.

Hence, $V^2 = \begin{vmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{vmatrix}$, etc.

16. The discriminating cubic is $\lambda^3 - 7\lambda + 6 = 0$ with roots -3, 1 and 2. The corresponding principal axes by (69.1) are in the directions $\{1, -1, -\sqrt{6}\}$, $\{1, 1, 0\}$ and $\{\sqrt{3}, -\sqrt{3}, \sqrt{2}\}$. The principal planes are obtained by using (68.1).

17. (*a*) The middle root of the discriminating cubic is 1 and so the real planes of circular section are parallel to $x^2 + y^2 + z^2 - \sqrt{6}yz + \sqrt{6}zx - 4xy \equiv \frac{1}{2}[\sqrt{2}x - \sqrt{2}(2 + \sqrt{3})y + (\sqrt{3} + 1)z][\sqrt{2}x - \sqrt{2}(2 - \sqrt{3})y + (\sqrt{3} - 1)z] = 0$. Hence, etc.

(*b*) The middle root of the discriminating cubic is $1/2$ and so the real planes of circular section are parallel to $x^2 + y^2 - z^2 + 2xy = 0$. Hence, etc.

18. In this example

$$\boldsymbol{\Delta} \equiv \begin{pmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha & 0 & 0 \\ -\sin\alpha\cos\alpha & \sin^2\alpha - \cos^2\alpha & -\sin\alpha\cos\alpha & 1 \\ 0 & -\sin\alpha\cos\alpha & -\sin^2\alpha & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \Delta = -\sin^2\alpha\cos^2\alpha,$$

and so $\Delta = 0$ for (*a*) $\alpha = n\pi$ and (*b*) $\alpha = (n + \frac{1}{2})\pi$. The corresponding matrices $\mathbf{\Delta}$ are respectively

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

These matrices are respectively of rank 2 and rank 3. Hence the corresponding degeneracies are the plane-pair $(x + y - 1)(x - y + 1) = 0$ and the hyperbolic cylinder $(y + 1)^2 - z^2 = 2$.

Miscellaneous examples

1. The enveloping cone from A to $z^2 + 2xy + 2az = 0$ is $y_A{}^2x^2 + x_A{}^2y^2 + (a^2 - 2x_Ay_A)z^2 + 2x_A(z_A + a)yz + 2ay_Azx - 2(z_A{}^2 + x_Ay_A + 2az_A)xy + 2ay_Az_Ax + 2az_Ax_Ay - 2a(2x_Ay_A + az_A)z + a^2z_A{}^2 = 0$. For the other enveloping cone, replace a by b. The two cones are identical if

$$\frac{y_A{}^2}{y_A{}^2} = \frac{x_A{}^2}{x_A{}^2} = \frac{a^2 - 2x_Ay_A}{b^2 - 2x_Ay_A} = \frac{x_A(z_A + a)}{x_A(z_A + b)} = \frac{ay_A}{by_A} = \frac{z_A{}^2 + x_Ay_A + 2az_A}{z_A{}^2 + x_Ay_A + 2bz_A}$$

$$= \frac{ay_Az_A}{by_Az_A} = \frac{az_Ax_A}{bz_Ax_A} = \frac{2x_Ay_A + az_A}{2x_Ay_A + bz_A} = \frac{a^2z_A{}^2}{b^2z_A{}^2}.$$

These equations are satisfied by $x_A = y_A = 0$ and $a^2/b^2 = (z_A{}^2 + 2az_A)/(z_A{}^2 + 2bz_A)$ from which $z_A = -2ab/(a + b)$. Hence, etc.

2. With the notation of Example 13, the equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ \alpha_1l_1 & \alpha_1m_1 & \alpha_1n_1 & 1 \\ \alpha_2l_2 & \alpha_2m_2 & \alpha_2n_2 & 1 \\ \alpha_3l_3 & \alpha_3m_3 & \alpha_3n_3 & 1 \end{vmatrix} = 0. \text{ That is, } \begin{vmatrix} x & y & z & 1 \\ a^2 & 0 & 0 & u_1 \\ 0 & b^2 & 0 & u_2 \\ 0 & 0 & c^2 & u_3 \end{vmatrix} = 0$$

where $u_1 = \alpha_1l_1 + \alpha_2l_2 + \alpha_3l_3$, $u_2 = \alpha_1m_1 + \alpha_2m_2 + \alpha_3m_3$ and $u_3 = \alpha_1n_1 + \alpha_2n_2 + \alpha_3n_3$. This equation simplifies to $u_1x/a^2 + u_2y/b^2 + u_3z/c^2 = 1$. Its pole is at the point (u_1, u_2, u_3). Hence, etc.

3. The straight line $(x - x_A)/l =$ etc. is a generator if $\Sigma f(y_A + mt)(z_A + nt) + fgh \equiv 0$. Thus $fmn + gnl + hlm = 0$ and $\Sigma f(ny_A + mz_A) = 0$. There are two solutions for $l:m:n$ and required condition is the same that the cone $fyz + gzx + hzy = 0$ is cut in orthogonal generators by the plane $\Sigma f(zy_A + yz_A) = 0$. From §29, the condition is $\Sigma f(fz_A + hx_A)(gx_A + fy_A) = 0$. That is, $fgh\Sigma x_A{}^2 + \Sigma f^2\Sigma fy_Az_A = 0$. Hence, etc.

4. The circle is represented by $x^2 + y^2 + z^2 - a^2 = x + y + z - a = 0$ and so the quadric is $x^2 + y^2 + z^2 - a^2 + (x + y + z - a)(\lambda x + \mu y + \nu z + \rho) = 0$. The line $y = z = 0$ is a generator and so $x^2 - a^2 + (x - a)(\lambda x + \rho) \equiv 0$ from which $\lambda = -1$ and $\rho = -a$. The line $z = x = 0$ is a generator and so similarly $\mu = -1$. Hence the quadric is $(1 + \nu)z^2 + (\nu - 1)yz + (\nu - 1)zx - 2xy - a(1 + \nu)z = 0$. The equations determining the centre are $-2y + (\nu - 1)z = -2x + (\nu - 1)z = 2(1 + \nu)z + (\nu - 1)y + (\nu - 1)x - a(1 + \nu) = 0$. Elimination of ν yields the result.

5. The roots of the discriminating cubic are 0, 25, 50 and 0, 9, 50. Thus 25 and 9 give the respective real planes of circular section when substituted for λ in (71.1). The required plane is $4x - 3y + 5z = 0$ whilst the square of the radius is $x^2 + y^2 + z^2 = [x^2 + y^2 + (4x - 3y)^2]/25 = 1$ in virtue of the equation of the first cylinder.

6. The discriminating cubic is $\lambda^3 - (2 + f^2)\lambda + 2f = 0$ with the roots f, $(-f \pm \sqrt{f^2 + 8})/2$. The middle root is f. Hence, etc.
When $f = 1$, the roots are 1, 1, -1 and so surface is of revolution.

7. The centre of the quadric is given by $y + z = z + x = x + y$ and so is at (0, 0, 0). Let $x/l = y/m = z/n$ be the diameter conjugate to $\lambda x + \mu y + \nu z + \rho = 0$. Thus this equation is the same as $l(y + z) + m(z + x) + n(x + y) = 0$. Comparing coefficients, $(m + n)/\lambda = (n + l)/\mu = (l + m)/\nu$ and so $l:m:n = \mu + \nu - \lambda:\nu + \lambda - \mu:\lambda + \mu - \nu$. The centre of the section is at the intersection of the plane and its conjugate diameter. This point is at infinity if $\lambda(\mu + \nu - \lambda) + \mu(\nu + \lambda - \mu) + \nu(\lambda + \mu - \nu) = 0$. Hence, etc.

8. The axes can be chosen so that the three skew lines are $z = x = 0$; $y = \lambda x, z = k$; $y = \mu x + \rho, z = l$. The latter two straight lines are generators of the quadric $\alpha(y - \lambda x)(y - \mu x - \rho) + \beta(y - \lambda x)(z - l) + \gamma(y - \mu x - \rho)(z - k) + \delta(z - k)(z - l) = 0$. The condition that $z = x = 0$ is also a generator yields $\alpha = 0$, $\beta = k$, $\gamma = -l$ and $\delta = \rho$. The quadric can be written in the form $k(y - \lambda x)(z - l) - (z - k)(ly - l\mu x - \rho z) = 0$ and so the other set of mutually skew generators are $k(y - \lambda x) = \theta(z - k)$, $\theta(z - l) = ly - l\mu x - \rho z$ where θ varies. For all values of θ, these generators are parallel to the plane whose direction-ratios are $\{k\lambda - l\mu, l - k, 0\}$.

9. Let the pencils be given by $u_1 + k_1v_1 = 0$, $u_2 + k_2v_2 = 0$. The homographic correspondence is $\alpha k_1k_2 + \beta k_1 + \gamma k_2 + \delta = 0$. Elimination of k_1 and k_2 yields $\alpha u_1u_2 - \beta u_1v_2 - \gamma u_2v_1 + \delta v_1v_2 = 0$. This is a quadric with $u_1 = v_1 = 0$ and $u_2 = v_2 = 0$ generators.

Chapter IX

1. The transformed equation $F(\bar{x}, \bar{y}, \bar{z}) = 0$ is a cone vertex at the new origin if $\bar{p} = \bar{q} = \bar{r} = \bar{d} = 0$. That is $\bar{p} = \bar{q} = \bar{r} = p\alpha + q\beta + r\gamma + d = 0$. Elimination of α, β, γ and δ yields the result.

2. From (75.2) we have $l_1 = k(m_2n_3 - m_3n_2)$, etc. Consequently

$$k\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = l_1^2 + m_1^2 + n_1^2 = 1. \text{ Thus } k = 1.$$

Alternatively: this result follows by considering vector products.

3. The inverse transformation is $x = \frac{1}{3}(\bar{x} + 2\bar{y} + 2\bar{z})$, $y = \frac{1}{3}(-2\bar{x} - \bar{y} + 2\bar{z})$, $z = \frac{1}{3}(2\bar{x} - 2\bar{y} + \bar{z})$. Hence, etc.

Chapter X

1. Apply the rotation $x = (\bar{x} - \bar{y})/\sqrt{2}$, $y = (\bar{x} + \bar{y})/\sqrt{2}$ and the equation becomes $3\bar{x}^2 + \bar{y}^2 + 2\bar{z}^2 + 2\sqrt{2}\bar{z}\bar{x} + 2\sqrt{2}\bar{x} + 2\bar{z} - 2 = 0$. Next rotate about the y-axis through an angle $\frac{1}{2}\tan^{-1} 2\sqrt{2}$ by means of the equations $\bar{x} = (\sqrt{2}x' - z')/\sqrt{3}$, $\bar{z} = (x' + \sqrt{2}z')/\sqrt{3}$ to obtain $4x'^2 + y'^2 + z'^2 + 2\sqrt{3}x'$

$-2 = 0$. Finally, translation to parallel axes through $(-\sqrt{3}/4, 0, 0)$ yields the result.

2. In these examples, the ranks of $\mathbf{\Delta}$ and $\mathbf{D}$ must be calculated. (See appendix, p. 116). Care must be taken to ensure that f, g, h, p, q, r are *half* the respective coefficients. Note carefully that the rank of $\mathbf{D}$ can be calculated simultaneously as that of $\mathbf{\Delta}$ if no multiples of the fourth row or column are added to the first three rows or columns.

6. The reader is recommended to work this example by the method of the section, rather than quote (83.2).

Miscellaneous examples

1. Select axes as in Fig. 10, §17. The locus is given by $(x \cos \alpha - y \sin \alpha)^2 + (z + c)^2 = (x \cos \alpha + y \sin \alpha)^2 + (z - c)^2$. That is $xy \sin \alpha \cos \alpha - cz = 0$. (**Alternatively,** consider the surface as the locus of the centres of spheres which touch the two straight lines). Calculation, yields $R(\mathbf{\Delta}) = 4$, $R(\mathbf{D}) = 2$.

2. Select the given generator as the x-axis. The quadric is then $by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2qy + 2rz = 0$. Next choose the normal at the origin as the z-axis. Now, the direction-ratios of the normal at P are $\{hy_P + gz_P,\ hx_P + by_P + fz_P + q,\ gx_P + fy_P + cz_P + r\}$ and so at the origin they are $\{0, q, r\}$. Hence $q = 0$. The normal at $(\alpha, 0, 0)$ on the generator has the equations $(x - \alpha)/0 = y/h\alpha = z/(g\alpha + r)$. Elimination of α yields the quadric $gxy - hzx + ry = 0$. Calculation shows that $R(\mathbf{\Delta}) = 4$, $R(\mathbf{D}) = 2$ and the two non-zero roots of the discriminating cubic are equal but opposite in sign.

3. In this example $P = 0$, $Q = \cos \alpha \sin \alpha$, $R = -\cos^2 \alpha$ and so equations (81.2) are $x/0 = \cos \alpha(y \cos \alpha + z \sin \alpha)/\cos \alpha \sin \alpha = [\sin \alpha\,(y \cos \alpha + z \sin \alpha) + 1]/-\cos^2 \alpha$. Hence, etc. The invariants are $I = 2$, $J = 1$, $D = 0$, $\Delta = -\cos^2 \alpha$ and so the reduced equation is obtained from (81.1).

4. The quadric with given generators is $\alpha xy + \beta x(z - k) + \gamma y(z + k) + \delta(z^2 - k^2) = 0$. Since it passes through the origin, $\delta = 0$. The quadric is of revolution if $R\begin{pmatrix} -\lambda & \alpha/2 & \beta/2 \\ \alpha/2 & -\lambda & \gamma/2 \\ \beta/2 & \gamma/2 & -\lambda \end{pmatrix} = 1$. That is, $\alpha\beta/\gamma = \beta\gamma/\alpha = \gamma\alpha/\beta$ and so $\alpha^2 = \beta^2 = \gamma^2$. The invariants are $I = 0$, $J = -3\alpha^2/4$, $D = \pm\, \alpha^3/4$, $\Delta = \alpha^4 k^2/4$. The roots of the discriminating cubic are $-\alpha/2, -\alpha/2, \alpha$ or $\alpha/2, \alpha/2, -\alpha$. Hence, etc.

5. Let the circles be $y = x^2 + z^2 - 2az = 0$ and $z = x^2 + y^2 - 2ay = 0$. A quadric through both circles is given by $x^2 + y^2 + z^2 + 2fyz - 2a(y + z) = 0$. We have $\Delta = 2a^2(f - 1)$ and $D = 1 - f^2$ and so $f = -1$ corresponds to a paraboloid and $f = 1$ to a cylinder. In both cases the non-zero roots of the discriminating cubic are 2 and 1. For $f = 1$, note that $A = \mathscr{A} = 0$ but $B/\mathscr{B} = -a^2$. Hence, etc.

6. The reduced equation is $2x^2 - y^2 - z^2 = 0$. Apply Example 5, §28.

7. Calculation yields $\Delta = 0$, $A = B = C = F = G = H = -(\beta\gamma + \gamma\alpha + \alpha\beta)$, $P = Q = R = D = 0$, $\mathscr{A} = \mathscr{B} = \mathscr{C} = \mathscr{F} = \mathscr{G} = \mathscr{H} = \beta\gamma + \gamma\alpha + \alpha\beta$. If $\beta\gamma + \gamma\alpha + \alpha\beta \neq 0$, $R(\mathbf{\Delta}) = 3$, $R(\mathbf{D}) = 2$ and so the quadric is a cylinder whose reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 = 1$. Further, $I = 2(\alpha + \beta + \gamma)$ and $J = 3(\beta\gamma + \gamma\alpha + \alpha\beta)$. Consequently λ_1, λ_2 are the roots of λ^2

$-2(\alpha + \beta + \gamma)\lambda + 3(\beta\gamma + \gamma\alpha + \alpha\beta) = 0$. The cylinder is hyperbolic if $\lambda_1\lambda_2 < 0$ but elliptic if $\lambda_1\lambda_2 > 0$ and $\lambda_1 + \lambda_2 > 0$.

If $\beta\gamma + \gamma\alpha + \alpha\beta = 0$, then $R(\mathbf{\Delta}) = 2$, $R(\mathbf{D}) = 1$ and so quadric is a parallel plane-pair.

8. Calculation yields $\Delta = -D = a^3 + b^3 + c^3 - 3abc \equiv \frac{1}{2}(a + b + c)[(b - c)^2 + (c - a)^2 + (a - b)^2]$. Thus for $a + b + c \neq 0$ we have $R(\mathbf{\Delta}) = 4$ and $R(\mathbf{D}) = 3$. Further, $I = a + b + c$, $J = bc + ca + ab - a^2 - b^2 - c^2$ and so the roots of the discriminating are $a + b + c$, $\pm\sqrt{a^2 + b^2 + c^2 - bc - ca - ab}$. The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 1$. Hence, etc.

If $a + b + c = 0$, calculation yields $R(\mathbf{\Delta}) = 3$ and $R(\mathbf{D}) = 2$ and so quadric is a cylinder which is hyperbolic with the reduced equation $\sqrt{bc + ca + ab}\,(x^2 - y^2) = 1$.

Chapter XI

1. The locus is $(x - x_K)^2 + (y - y_K)^2 + (z - z_K)^2 = (\lambda x + \mu y + \nu z + \rho)^2$, where $[\lambda, \mu, \nu]$ are direction-cosines. Calculation yields $D = 0$. Hence, etc.

2. Result follows from the identity $x^2/A + y^2/B + z^2/C - 1 \equiv \{(x - x_K)^2 + (y - y_K)^2 + z^2\}/C + (1/A - 1/C)\{x - Ax_K/(A - C)\}^2 + (1/B - 1/C)\{y + By_K/(C - B)\}^2$ when $z_K = 0$ and $y_K^2/(B - C) - x_K^2/(C - A) = 1$.

3. If $a = b$, the rank of the matrix at the beginning of this section is 2 for $\lambda = a$ or c. The line $x = y = 0$ corresponds to a whilst the circle corresponds to c.

4. The focal ellipse and hyperbola for $A > B > C$ are (85.3) and (85.2) respectively. Their real foci are at $(\pm\sqrt{A - B}, 0, 0)$ and $(0, \pm\sqrt{A - C}, 0.)$ Hence, etc.

5. For a cone $d = 0$ and so division by d is not allowable. The equations (85.1), (85.2) and (85.3) with zero right-hand sides give three pairs of focal lines of which only one pair is real.

6. When $a = b$ the rank of the matrix at the beginning of this section is 2 only for $\lambda = a$. Focal conics reduce to $x = y = 0$ taken twice.

7. $(x - x_P)/l = (y - y_P)/m = (z - z_P)/n$ touches (87.1) if $(\Sigma x_P l/(A - k))^2 = (\Sigma l^2/(A - k))(\Sigma x_P^2/(A - k) - 1)$. (Example 2, p. 47.) By Lagrange's identity, this equation may be written $\Sigma(ny_P - mz_P)^2/(B - k)(C - k) = \Sigma l^2/(A - k)$. This is a quadratic in k and so *two* confocals corresponding to k_1 and k_2 touch the line at Q and R (say). Since R lies in the tangent plane at Q we have $\Sigma x_Q x_R/(A - k_1) = 0$. Similarly, $\Sigma x_R x_Q/(A - k_2) = 0$. By subtraction and division by $k_1 - k_2$ we have $\Sigma x_Q x_R/(A - k_1)(A - k_2) = 0$. Hence, etc.

8. The pole of $\lambda x + \mu y + \nu z + \rho = 0$ with respect to (87.1) is at $x = -\lambda(A - k)/\rho$, $y = -\mu(B - k)/\rho$, $z = -\nu(C - k)/\rho$ and so $x\rho/\lambda + A = y\rho/\mu + B = z\rho/\nu + C$ which represents a straight line orthogonal to the direction $\{\lambda, \mu, \nu\}$. Further, the point of contact of the given plane with a confocal of the system is on the locus. Hence, etc.

9. From the cubic equation in k of this section, we have $\Sigma k_1 = -\Sigma x_P^2 + \Sigma A$, $\Sigma k_2 k_3 = -\Sigma(B + C)x_P^2 + \Sigma BC$, $k_1 k_2 k_3 = \Sigma BCx_P^2 - ABC$. Hence, etc.

10. By (63.1) the plane $\lambda x + \mu y + \nu z + \rho = 0$ (ρ variable) touches the confocal (88.1) if $(A - k)\lambda^2 + (B - k)\mu^2 - k\nu^2 - 2\nu\rho = 0$. By (62.1) the point of contact is at $x = (k - A)\lambda/\nu$, $y = (k - B)\mu/\nu$, $z = (k\nu + \rho)/\nu$.

Elimination of k yields the straight-line $x\nu/\lambda + A = y\nu/\mu + B = (2\nu^2 z - A\lambda^2 - B\mu^2)/(\nu^2 - \lambda^2 - \mu^2)$. The point $(0, \mu(A - B)/\nu, [A\nu^2 - (B - A)\mu^2]/2\nu^2)$ lies on this plane and one of the focal parabolae. Hence, etc.

Miscellaneous examples

1. $R \begin{pmatrix} 1 - \lambda & 0 & 0 & \lambda x_K \\ 0 & -\lambda & 4 & \lambda y_K \\ 0 & 4 & -\lambda & \lambda z_K \\ \lambda x_K & \lambda y_K & \lambda z_K & -1 - \lambda(x_K{}^2 + y_K{}^2 + z_K{}^2) \end{pmatrix} = 2$ when $\lambda = 1$, 4 or $-$ 4.

$\lambda = 1$ yields the virtual conic $16(y^2 + z^2) + 8yz = -15$, $x = 0$. $\lambda = 4$ yields the focal hyperbola and $\lambda = -4$ the focal ellipse.

2. See Examples 2 and 3, p. 63.

3. $x_L/a = x_{L'}/\sqrt{a^2 - k}$, etc. Thus $(x_L - x_{M'})^2 - (x_{L'} - x_M)^2 = k(x_L{}^2 - x_M{}^2)/a^2$ and so $LM'^2 - L'M^2 = k[\Sigma x_L{}^2/a^2 - \Sigma x_M{}^2/a^2] = 0$. Hence, etc.

4. The normals from K to the quadric (87.1) lie on the quadric cone $\Sigma(C - B)x_K(y - y_K)(z - z_K) = 0$ by (40.5).

Further, equation (48.5) shows that the normals from K to the quadric (88.1) lie on the quadric cone $(z - z_K)(x_K y - y_K x) + (A - B)(x - x_K)(y - y_K) = 0$. (The presence of $-k$ in the right-hand side of (88.1) has no effect on the result.)

5. The normal at P to the confocal (87.1) with $k = k_1$ has the equations $(A - k_1)(x - x_P)/x_P = (B - k_1)(y - y_P)/y_P = (C - k_1)(z - z_P)/z_P$. Let this normal cut the polar plane of P with respect to $x^2/A + y^2/B + z^2/C = 1$ at Q. Then $Q \equiv (x_P[1 + t/(A - k_1)], y_P[1 + t/(B - k_1)], z_P[1 + t/(C - k_1)])$. But Q lies on $x_P x/A + y_P y/B + z_P z/C = 1$ and so $\Sigma \dfrac{x_P{}^2}{A}\left(1 + \dfrac{t}{A - k_1}\right) = 1 = \Sigma \dfrac{x_P{}^2}{A - k_1}$. Hence $t = k_1$ and $Q \equiv (Ax_P/(A - k_1), By_P/(B - k_1), Cz_P/(C - k_1))$. The polar plane of Q with respect to F is $\Sigma x_P x/(A - k_1) = 1$ which is the tangent plane at P to the confocal $k = k_1$. Hence, etc.

6. Let P be the vertex of the cone and Q, R, S be the points where the normals at P to the confocals through P intersect the polar of P. (The polar plane of P with respect to the tangent cone is also the polar plane of P with respect to the quadric.) By the previous example $PQRS$ is a self-polar tetrahedron with respect to the quadric. Hence QRS is a self-polar triangle of the section of the quadric by the polar plane of P. Thus PQ, PR, PS are conjugate diameters of the cone and since they are mutually orthogonal they are the principal axes.

7. The polar line of PQ with respect to (87.1) is the line of intersection of the planes $\Sigma x_P x/(A - k) = 1$ and $\Sigma x_Q x/(A - k) = 1$ and so has direction-ratios $(A - k)(y_P z_Q - y_Q z_P)$, etc. The orthogonality condition is $\Sigma(A - k)(x_P - x_Q)(y_P z_Q - y_Q z_P) = 0$. That is $\Sigma A(x_P - x_Q)(y_P z_Q - y_Q z_P) = 0$, which is independent of k. Hence, etc.

8. The discriminating cubics are $\lambda^3 - \alpha\lambda - 2\beta = 0$ and $\lambda'^3 - \alpha'\lambda' - 2\beta' = 0$, where $\alpha = f^2 + g^2 + h^2$, $\beta = fgh$, $\alpha' = f'^2 + g'^2 + h'^2$ and $\beta' = f'g'h'$. If the two quadrics are confocal, $1/\lambda' = 1/\lambda + k$ and so $\lambda' = \lambda/(1 + k\lambda)$. Thus $\lambda^3 - \alpha'(1 + k\lambda)^2\lambda - 2\beta'(1 + k\lambda)^3 = 0$ is the same equation as $\lambda^3 - \alpha\lambda - 2\beta = 0$. Comparing coefficients, we have $2\alpha'k + 6\beta'k^2 = 0$ and $1 - \alpha'k^2 - 2\beta'k^3 = (\alpha' + 6\beta'k)/\alpha = \beta'/\beta$. Hence, etc.

Chapter XII

1. The polar plane of A with respect to (89.1) is $F_A + kF'_A = 0$ which forms the pencil of planes with the straight line $F_A = F'_A = 0$ as axis.

2. The polar line of AB with respect to (89.1) is $F_A + kF'_A = F_B + kF'_B = 0$ and so it generates the quadric $F_AF'_B - F_BF'_A = 0$.

3. Consider the matrices $\mathbf{\Delta} \equiv \begin{pmatrix} 1 & 0 & 1-\frac{1}{2}k & \frac{1}{2}k \\ 0 & 1+k & 0 & 0 \\ 1-\frac{1}{2}k & 0 & 0 & 0 \\ \frac{1}{2}k & 0 & 0 & 0 \end{pmatrix}$ and

$\mathbf{D} \equiv \begin{pmatrix} 1 & 0 & 1-\frac{1}{2}k \\ 0 & 1+k & 0 \\ 1-\frac{1}{2}k & 0 & 0 \end{pmatrix}$. In general, $R(\mathbf{\Delta}) = 3$ and $R(\mathbf{D}) = 3$.

For $k = 2$, we have $R(\mathbf{\Delta}) = 3$ and $R(\mathbf{D}) = 2$ which corresponds to a cylinder whose reduced equation is $x^2 + 3y^2 = 1$. For $k = -1$, we have $R(\mathbf{\Delta}) = R(\mathbf{D}) = 2$ which corresponds to a plane-pair.

4. A transformation of axes exists for which the plane $u = 0$ becomes the plane $x = 0$ and the plane $v = 0$ becomes the plane $\lambda x + \mu y = 0$ or $\lambda x + \rho = 0$ according as $u = 0$ and $v = 0$ intersect or are parallel. In either case, equation (90.1) is quadratic in k. Hence, etc.

5. Let the quadrics be $F_1 \equiv F - k_1u_1^2 = 0$ and $F_2 \equiv F - k_2u_2^2 = 0$. Then $F_1 \equiv F_2 + kuv$ where $kuv = k_2u_2^2 - k_1u_1^2$. Hence, etc.

6. (i) If F and F' represent the quadrics, equation (90.1) has the repeated root $k = 1$ which corresponds to the coincident pair of planes $(x + y + z + 1)^2 = 0$. Thus the quadrics have ring contact.

(ii) In this case, equation (90.1) has a simple root $k = -1$ which corresponds to $(x - y - 1)(x - z + 1) = 0$ and so represents a plane-pair. Thus the quadrics have double contact.

7. The polar plane of A with respect to (92.1) is $F_A + k'F'_A + k''F''_A = 0$ which passes through the fixed point $F_A = F'_A = F''_A = 0$.

8. The three degenerate quadrics $A_1A_2A_4B_3$, $B_1B_2B_4A_3$; $A_2A_3A_4B_1$, $B_2B_3B_4A_1$; $A_3A_1A_4B_2$, $B_3B_1B_4A_2$ pass through the eight points A_i, B_i and so they form a set of associated points. A fourth degenerate quadric through seven of these points in the plane-pair $A_1A_2A_3B_4$, $B_1B_2B_3$. Hence A_4 lies on this quadric, etc.

Miscellaneous examples

1. All quadrics through the parabola are represented by $x^2 - 4ay - a^2 + z(\lambda x + \mu y + \nu z + \rho) = 0$. The section by $y = 0$ is $x^2 + z^2 - a^2 = 0$ and so $\nu = 1$, $\lambda = \rho = 0$. Hence the pencil has the equation $x^2 + z^2 + \mu yz - 4ay - a^2 = 0$. For this pencil $\Delta = -4a^2 + a^2\mu^2/4$ and $D = -\mu^2/4$ and so cones correspond to $\mu = \pm 4$ and a paraboloid to $\mu = 0$.

The centres of the quadrics of the pencil satisfy $x = 0$, $-4a + \mu z = 0$ and $2z + \mu y = 0$. Hence, etc.

2. Equation (90.1) for the pencil $F + k(\lambda x + \mu y + \nu z + \rho)^2 = 0$ reduces to $\Delta + k\Sigma = 0$ which is linear in k. (See equation (95.2) for the notation Σ.)

The cone degenerates to a cylinder if $\begin{vmatrix} a + k\lambda^2 & h + k\lambda\mu & g + k\lambda\nu \\ h + k\lambda\mu & b + k\mu^2 & f + k\mu\nu \\ g + k\lambda\nu & f + k\mu\nu & c + k\nu^2 \end{vmatrix} = 0.$

This equation reduces to $D + k\phi = 0$. (See equation (29.2).) Hence the required condition is $\Delta\phi = D\Sigma$.

3. From the matrix $\mathbf{\Delta}$ of the pencil $x^2 + y^2 + 2z^2 + 2az + k(z^2 - 2xy) = 0$ we calculate that $\Delta = a^2(k^2 - 1)$ and $D = (k + 2)(1 - k^2)$. Thus, cylinders correspond to $k = \pm 1$. Hence, etc.

4. Choose the tangent plane at the origin to be $z = 0$ and so the sphere is $x^2 + y^2 + z^2 + 2rz = 0$. A quadric having ring contact is $x^2 + y^2 + z^2 + 2rz + k(\lambda x + \mu y + \nu z + \rho)^2 = 0$. Its section by $z = 0$ is $x^2 + y^2 + k(\lambda x + \mu y + \rho)^2 = 0$ which clearly represents a conic with focus at the origin and directrix $\lambda x + \mu y + \rho = 0$.

5. Let $u_i \equiv \lambda_i x + \mu_i y + \nu_i z + \rho_i$. Then the polar plane of A with respect to $\Sigma k_i u_i^2 = 0$ has the equation $\Sigma k_i(\lambda_i x_A + \mu_i y_A + \nu_i z_A + \rho_i)u_i = 0$. Let A be the point $u_2 = u_3 = u_4 = 0$. Then the coefficients of u_2, u_3, u_4 in the above equation are zero and so the polar plane is $u_1 = 0$.

6. The equation $k_1u_1v_1 + k_2u_2v_2 + k_3u_3v_3 = 0$ is satisfied by the eight points corresponding to $u_1 = u_2 = u_3 = 0$, $u_1 = u_2 = v_3 = 0$, $u_1 = v_2 = u_3 = 0$, $u_1 = v_2 = v_3 = 0$, $v_1 = u_2 = u_3 = 0$, $v_1 = u_2 = v_3 = 0$, $v_1 = v_2 = u_3 = 0$ and $v_1 = v_2 = v_3 = 0$.

Seven vertices of a hexahedron determine the hexahedron. Hence, etc.

7. The point (x, y, z) on the cone has coordinates $x = x_A(1 + k) - x_U$ where A is the vertex and U any point on $F = u = 0$. That is $F_U = 0, u_U = 0$. Hence $F = (1 + k)^2F_A - 2(1 + k)T_{AU}$, $T_A = (1 + k)F_A - T_{AU}$, $u = (1 + k)u_A$. Elimination of T_{AU} and $1 + k$ yields the result.

The equation of the cone can be written $u_A^2F + u(F_Au - 2u_AT_A) = 0$. Hence, etc.

Chapter XIII

2. (i) $\begin{vmatrix} A & H & G & P \\ H & B & F & Q \\ G & F & C & R \\ P & Q & R & D \end{vmatrix} \times \begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix} \equiv \begin{vmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{vmatrix}$. Hence, etc.

(ii) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ H & B & F & Q \\ G & F & C & R \\ P & Q & R & D \end{vmatrix} \times \begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix} \equiv \begin{vmatrix} a & h & g & p \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{vmatrix}$. Hence, etc.

3. (i) $\Delta \equiv \begin{pmatrix} 0 & \frac{1}{2}\gamma & \frac{1}{2}\beta & 0 \\ \frac{1}{2}\gamma & 0 & \frac{1}{2}\alpha & 0 \\ \frac{1}{2}\beta & \frac{1}{2}\alpha & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and so

$$\nabla \equiv \begin{pmatrix} \frac{1}{4}\alpha^2 & -\frac{1}{4}\alpha\beta & -\frac{1}{4}\gamma\alpha & 0 \\ -\frac{1}{4}\alpha\beta & \frac{1}{4}\beta^2 & -\frac{1}{4}\beta\gamma & 0 \\ -\frac{1}{4}\gamma\alpha & -\frac{1}{4}\beta\gamma & \frac{1}{4}\gamma^2 & 0 \\ 0 & 0 & 0 & \frac{1}{4}\alpha\beta\gamma \end{pmatrix}$$

(ii) $\Delta \equiv \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and so $\nabla \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$

(iii) $\Delta \equiv \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and so $\nabla \equiv \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

4. $\nabla = 0$ yields $k = -1$. The equation $\lambda^2 + \mu^2 - \nu^2 + 4\lambda\mu - 2\lambda\nu + 2\mu\rho - \rho^2 = 0$ represents a conic whose plane is given by $2\lambda + 4\mu - 2\nu = 4\lambda + 2\mu + 2\rho = -2\lambda - 2\nu = 0$. That is, $\lambda = -\mu = -\nu = -\rho$ and so plane is $x - y - z - 1 = 0$. The cylinder enveloped by all the tangent planes parallel to the z-axis satisfies $\lambda^2 + \mu^2 + 4\lambda\mu + 2\mu\rho - \rho^2 = 0$ which has the point equation $2x^2 + y^2 - 4xy - 4x + 2y + 3 = 0$.

5. The point equation corresponding to $\lambda^2 + 2\mu^2 + \rho^2 + 4\mu\nu = 0$ is $F \equiv 2x^2 - z^2 + 2yz + 2 = 0$. The result follows from (97.2) with $A \equiv (-1, -2, 0)$.

6. (i) The tangential equation corresponding to $x^2 + y^2 + 2zx + 2xy + 3 = 0$ is $\Sigma \equiv 3\mu^2 + 6\lambda\nu - 6\mu\nu + \rho^2 = 0$. The result follows from (98.2) with $a \equiv (1, 2, 0, 0)$.

(ii) This problem needs care since $2x^2 + y^2 - 4xy - 4x + 2y + 3 = 0$ is degenerate and so cannot be expressed by a *single* tangential equation. The conic can also be expressed by the equations $x - y - z - 1 = 2x^2 + y^2 - 4xy - 2y - 4z - 1 = 0$. The quadric $2x^2 + y^2 - 4xy - 2y - 4z - 1 = 0$ is not degenerate and has the tangential equation $\lambda^2 + 2\mu^2 + 4\lambda\mu - 2\lambda\nu - 2\mu\nu + 2\nu\rho = 0$. The result now follows from (98.2) with $a \equiv (1, -1, -1, -1)$.

7. This result is dual to Example 2, p. 102.

8. The tangential equation is $4\lambda^2 + 2\mu\nu - 4\rho^2 = 0$. The values of k which make (100.1) degenerate are $\pm 1, -4$. The real focal conics correspond to ± 1.

Miscellaneous examples

1. The tangential equations of the sphere and hyperbola are $\lambda^2 + \mu^2 + \nu^2 - \rho^2 = 0$ and $\lambda^2 - \mu^2 + \nu^2 - 2\lambda\rho + \rho^2 = 0$. [See solution of **6** (ii).] The required conics are the degeneracies of $\lambda^2 - \mu^2 + \nu^2 - 2\lambda\rho + \rho^2 + k(\lambda^2 + \mu^2 + \nu^2 - \rho^2) = 0$. These correspond to $k = 0, \pm 1$. Hence, etc.

2. The focal parabolae are the degenerate conics of $\nu^2 + 4\nu\mu + 4\lambda\rho + 4\mu\rho + k(\lambda^2 + \mu^2 + \nu^2) = 0$. These correspond to $k = 1, -2$.

3. The focal conics are the degenerate conics of $\alpha(\lambda^2 + 2\mu\nu) + 2\nu\lambda + 2\lambda\mu + \rho^2 + k(\lambda^2 + \mu^2 + \nu^2) = 0$. The corresponding values of k are $\alpha, -\alpha \pm \sqrt{2}$. The conic corresponding to $k = \alpha$ is $2\alpha\lambda^2 + \alpha\mu^2 + \alpha\nu^2 + 2\alpha\mu\nu + 2\nu\lambda + 2\lambda\mu + \rho^2 = 0$ which lies in the plane determined by $2\alpha\lambda + \mu + \nu = \lambda + \alpha\mu + \alpha\nu = \rho = 0$. That is, $\lambda = 0, \mu = 1, \nu = -1, \rho = 0$ and so the plane is $y - z = 0$. The point-equation is $\alpha x^2 + 2\alpha y^2 - 2xy + 2\alpha^2 - 1 = y - z = 0$. Make the rotation corresponding to $x = x', y = (y' + z')/\sqrt{2}, z = (y' - z')/\sqrt{2}$ and dropping the primes, the equations become $\alpha x^2 + \alpha(y + z)^2 - \sqrt{2}x(y + z) + 2\alpha^2 - 1 = \sqrt{2}z = 0$. That is $\alpha(x^2 + y^2) - \sqrt{2}xy + 2\alpha^2 - 1 = z = 0$. The line-equation of this conic is $\alpha(\lambda^2 + \mu^2) + \sqrt{2}\lambda\mu + 1/2 = 0$ which represents a system of confocal conics.

4. The planes parallel to the x-axis are $\mu y + \nu z + \rho = 0$ and the point at infinity on the x-axis is $\lambda = 0$. Thus the required equations are $\lambda = B\mu^2 + C\nu^2 + D\rho^2 + 2F\mu\nu + 2Q\mu\rho + 2R\nu\rho = 0$.

5. This diameter is the line joining the pole of the plane at infinity and the pole of the plane $x = 0$. That is, the line joining the poles of $(0, 0, 0, 1)$ and $(1, 0, 0, 0)$. Hence, etc.

6. The quadric $F' \equiv F + kuv = 0$ has double contact with F such that $u = 0$ and $v = 0$ are the common conics of intersection of the two quadrics. Dually $\Sigma' \equiv \Sigma + kUV = 0$, where U and V are both linear in λ, μ, ν, ρ is the equation of a quadric envelope which has double contact with Σ and such that $U = 0, V = 0$ are the equations of the vertices of the two common tangent cones to the quadric envelopes. These vertices are on the x-axis and so $\Sigma' \equiv \Sigma + k(\alpha\gamma + \rho)(\alpha'\lambda + \rho)$. The condition that $\Sigma' = 0$ be a paraboloid is $D + k = 0$. The quadric envelope touches the z-axis and so some tangent plane $(\lambda, \mu, 0, 0)$ exists through the z-axis. Thus $A\lambda^2 + B\mu^2 + 2H\lambda\mu - D\alpha\alpha'\lambda^2 = 0$ has coincident roots. Accordingly $H^2 = B(A - D\alpha\alpha')$.

INDEX

Angle between two generators of a cone 31
Associated points 104
Axes of plane section of a quadric 45, 53

Base curve 101
Bundle of planes 15

Centre of plane section 44, 52
Centre of quadric 71
Circle at infinity 56
Circular cone 30
Circular sections 61, 63, 75
Classification of quadric 85
Coaxal spheres 26
Cofactor notation 66
Confocal quadrics 97, 112
Conjugate diameters 71
Conjugate lines 71
Conjugate points 21, 40, 50, 66
Contact of quadrics 103
Corresponding points on two quadrics 100

Diametral planes 71
Directrix 93
Discriminating cubic 74
Distance between two straight lines 16
Distance from a point to a plane 14
Distance from a point to a straight line 18

Elliptic coordinates 98
Enveloping cone 41, 50, 68
Enveloping cylinder 69

Focal conics 95, 96
Focus 93

Generators 31, 41, 51, 69

Intersection of three planes 58
Invariants 79, 82

Limiting points of two spheres 27

Normals 43

Orthocentric tetrahedron 28
Orthogonal cone 34
Orthoptic sphere 41

Parabolic coordinates 100
Parametric equations of
  hyperbolic paraboloid 51
  hyperboloid of one sheet 42
Pencil of planes 15
Pencil of quadrics 101
Polar of line 70
Pole and polar 23, 40, 50, 67, 111
Principal diametral planes 73

Quadric envelope 107

Range of quadrics 112
Reciprocal cone 33
Rectangular cone 34
Rotation of axes 80

Self-conjugate tetrahedron 71
Self-polar tetrahedron 71, 102

Tangential equations 107, 110, 111
Translation of axes 78

Umbilic 62

Vertex of paraboloid 89

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Line Co-ordinates
Generalized Homogeneous
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Worked-out Solutions (Appendix)
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